RECIPROCAL FUNCTIONS

Given the graph of $y = f(x)$, the graph of $y = \frac{1}{f(x)}$ is the **reciprocal function** of $f(x)$.

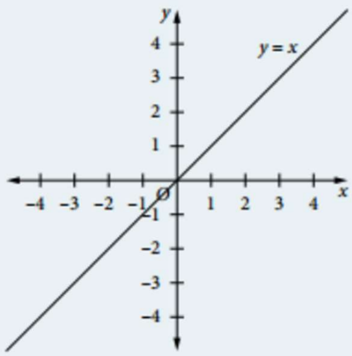
When graphing reciprocal functions, it is important to find where $f(x) = 0$, as these x values will give vertical asymptotes for the reciprocal function.

It is also important to note that where $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$.

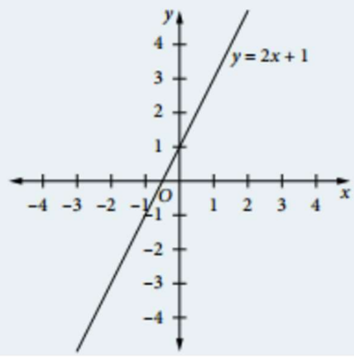
Example 1

In each part, use the graph of the given function to draw the graph of $y = \frac{1}{f(x)}$.

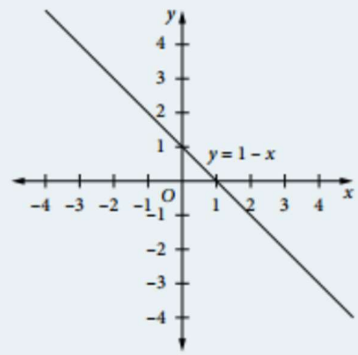
(a) Given $y = x$, draw $y = \frac{1}{x}$.



(b) Given $y = 2x + 1$, draw $y = \frac{1}{2x+1}$.

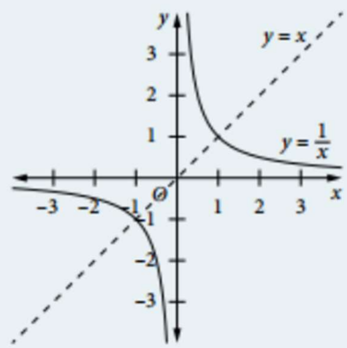


(c) Given $y = 1 - x$, draw $y = \frac{1}{1-x}$.

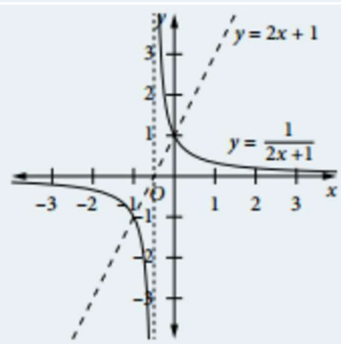


Solution

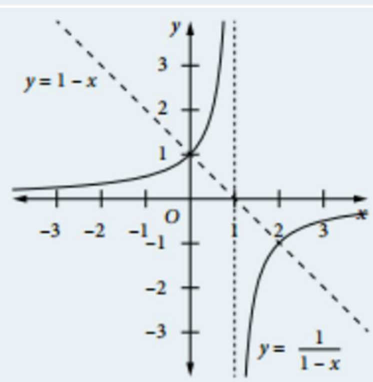
(a) The graph of $y = \frac{1}{x}$ is undefined at $x = 0$, so $x = 0$ is a vertical asymptote.
The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.
$y = 0$ is the horizontal asymptote.
$x = \frac{1}{x}$ where $x = \pm 1$, hence the curves intersect at $(-1, -1)$ and $(1, 1)$.



(b) The graph of $y = \frac{1}{2x+1}$ is undefined at $x = -\frac{1}{2}$, so $x = -\frac{1}{2}$ is a vertical asymptote.
The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.
$y = 0$ is the horizontal asymptote.
$2x + 1 = \frac{1}{2x+1}$ where $x = -1, 0$. Hence the curves intersect at $(-1, -1)$ and $(0, 1)$.



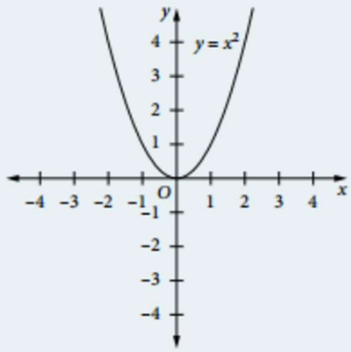
(c) The graph of $y = \frac{1}{1-x}$ is undefined at $x = 1$.
The graph approaches $y = 0$ from below as $x \to \infty$.
The graph approaches $y = 0$ from above as $x \to -\infty$.
$1 - x = \frac{1}{1-x}$ where $x = 0, 2$. The curves intersect at $(0, 1)$ and $(2, -1)$.
$x = 1$ is the vertical asymptote and $y = 0$ is the horizontal asymptote.



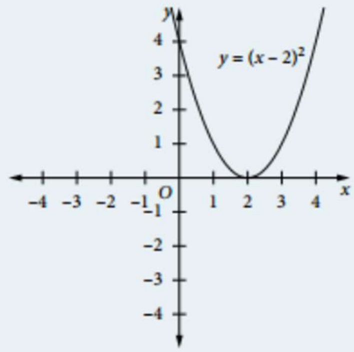
All these reciprocal functions are rectangular hyperbolas.

Example 2

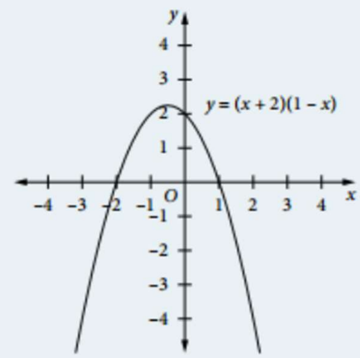
(a) Given the graph of $y = x^2$, draw $y = \frac{1}{x^2}$.



(b) Given the graph of $y = (x-2)^2$, draw $y = \frac{1}{(x-2)^2}$.



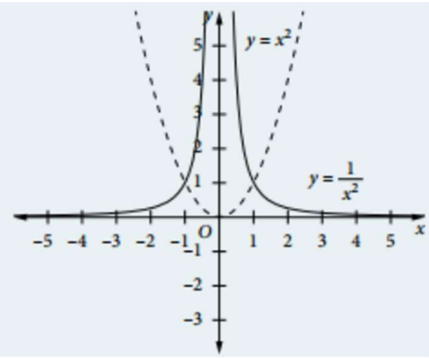
(c) Given the graph of $y = (x+2)(1-x)$, draw $y = \frac{1}{(x+2)(1-x)}$.



Solution

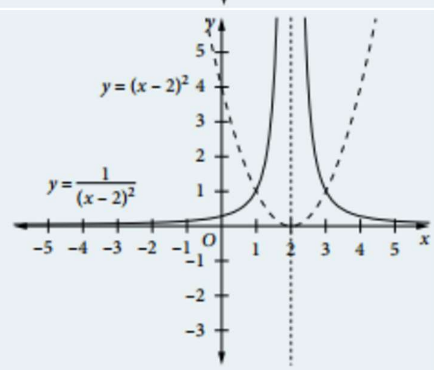
(a) The graph of $y = \frac{1}{x^2}$ is undefined at $x = 0$.

The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from above as $x \to -\infty$.
The function is never negative.
$x^2 = \frac{1}{x^2}$ where $x = \pm 1$. The curves intersect at $(-1, 1)$ and $(1, 1)$.
$x = 0$ is the vertical asymptote and $y = 0$ is the horizontal asymptote.



(b) The graph of $y = \frac{1}{(x-2)^2}$ is undefined at $x = 2$.

The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from above as $x \to -\infty$.
The function is never negative.
$(x-2)^2 = \frac{1}{(x-2)^2}$ where $x = 1, 3$. The curves intersect at $(1, 1)$ and $(3, 1)$.
$x = 2$ is the vertical asymptote and $y = 0$ is the horizontal asymptote.
$x = 2$ is an axis of symmetry.



(c) The graph of $y = \dfrac{1}{(x+2)(1-x)}$ is undefined at $x = -2, 1$.

The graph approaches $y = 0$ from below as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.

The maximum value of $(x+2)(1-x)$ is $\dfrac{9}{4}$ and occurs at $x = -\dfrac{1}{2}$.

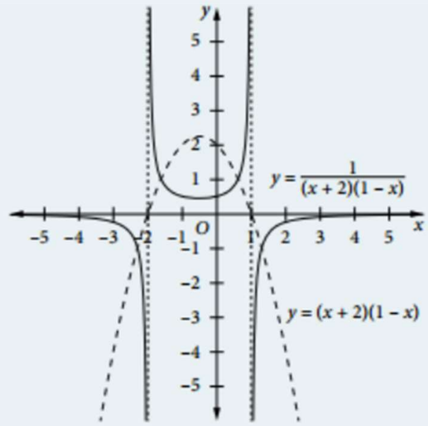
Thus the least positive value of $\dfrac{1}{(x+2)(1-x)}$ is $\dfrac{4}{9}$ and occurs at $x = -\dfrac{1}{2}$.

$x < -2$, $\dfrac{1}{(x+2)(1-x)} < 0$; $-2 < x < 1$, $\dfrac{1}{(x+2)(1-x)} > \dfrac{4}{9}$;

$x > 1$, $\dfrac{1}{(x+2)(1-x)} < 0$.

$x = -2$ and $x = 1$ are the vertical asymptotes, $y = 0$ is the horizontal asymptote.

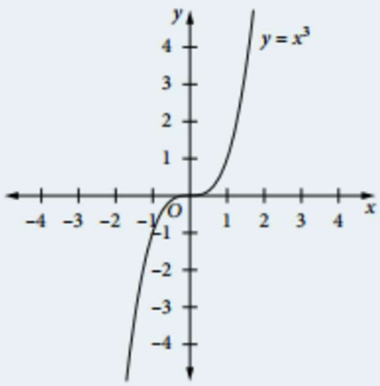
$x = -\dfrac{1}{2}$ is an axis of symmetry.



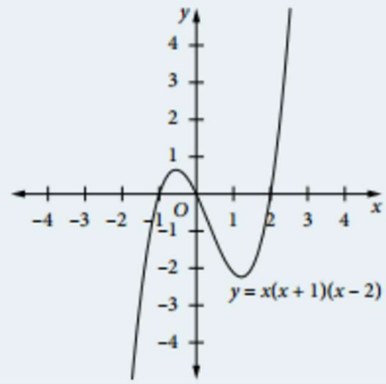
Each of these graphs has a vertical axis of symmetry.

Examples 3a) and 3b)

(a) Given the graph of $y = x^3$, draw $y = \frac{1}{x^3}$.



(b) Given the graph of $y = x(x+1)(x-2)$, draw the graph of $y = \frac{1}{x(x+1)(x-2)}$.



Solution

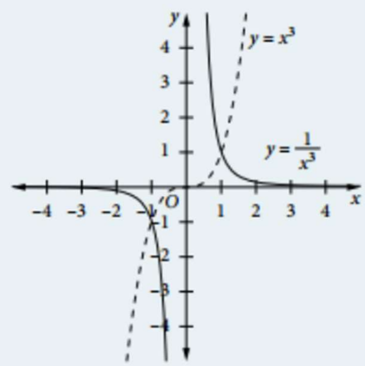
(a) The graph of $y = \frac{1}{x^3}$ is undefined at $x = 0$.

The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.

$x^3 = \frac{1}{x^3}$ where $x = \pm 1$. The curves intersect at $(-1, -1)$ and $(1, 1)$.

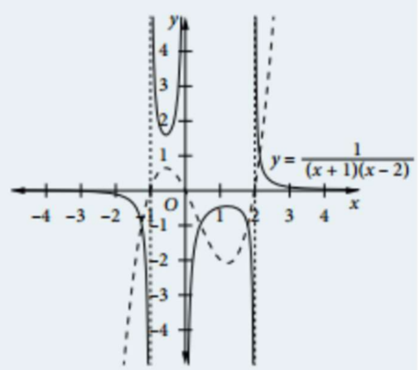
$x = 0$ is the vertical asymptote, $y = 0$ is the horizontal asymptote.

The curve does not have an axis of symmetry, but has rotational (point) symmetry about the origin.



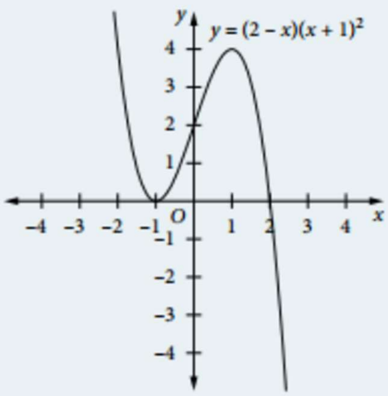
(b) The graph of $y = \frac{1}{x(x+1)(x-2)}$ is undefined at $x = -1, 0, 2$.

The graph approaches $y = 0$ from above as $x \to \infty$.
The graph approaches $y = 0$ from below as $x \to -\infty$.
$x = -1, 0, 2$ are the vertical asymptotes, $y = 0$ is the horizontal asymptote.
The curve does not have an axis of symmetry.

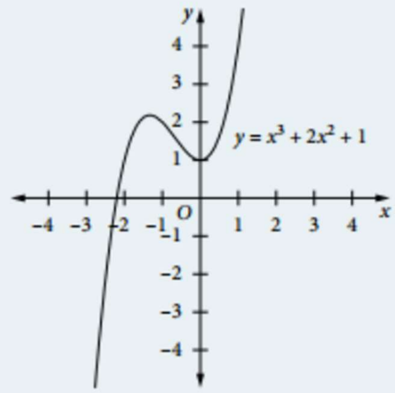


Examples 3c) and 3d)

(c) Given the graph of $y=(2-x)(x+1)^2$, draw the graph of $y=\dfrac{1}{(2-x)(x+1)^2}$.

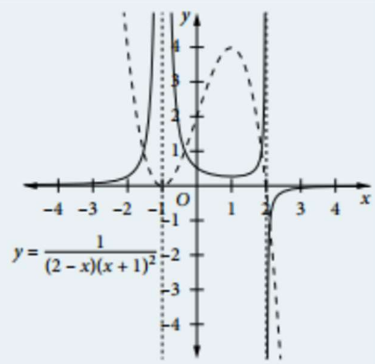


(d) Given the graph of $y=x^3+2x^2+1$, draw the graph of $y=\dfrac{1}{x^3+2x^2+1}$.

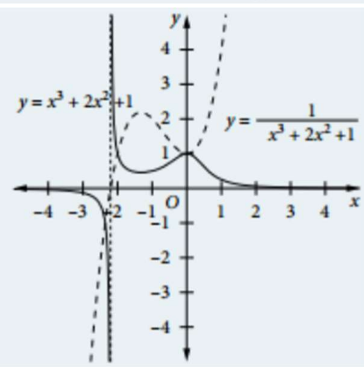


Solution

(c) The graph of $y=\dfrac{1}{(2-x)(x+1)^2}$ is undefined at $x=-1, 2$.
The graph approaches $y=0$ from below as $x\to\infty$.
The graph approaches $y=0$ from above as $x\to-\infty$.
$x=-1, 2$ are the vertical asymptotes, $y=0$ is the horizontal asymptote.
It looks as though $y=(2-x)(x+1)^2$ has a local maximum value of 4 at $x=1$. (This can be shown using calculus.)
Hence $y=\dfrac{1}{(2-x)(x+1)^2}$ will have a local minimum value of $\dfrac{1}{4}$ at $x=1$.



(d) The graph of $y=\dfrac{1}{x^3+2x^2+1}$ is undefined at $x\approx-2.2$.
The graph approaches $y=0$ from above as $x\to\infty$.
The graph approaches $y=0$ from below as $x\to-\infty$.
$x=-2.2$ is the vertical asymptote, $y=0$ is the horizontal asymptote.
The curves touch at (0, 1), a local minimum of the original function becomes a local maximum of the reciprocal function.



As the Examples above show, a maximum turning point on the original function becomes a minimum turning point on the reciprocal function (or equivalent asymptote). A minimum turning point on the original function becomes a maximum turning point on the reciprocal function (or equivalent asymptote).