Question 1: Prove by mathematical induction that for all positive integer values of n:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Step 1: When $n = 1$, the left-hand side (LHS) is $LHS = 1$, whereas the right-hand side (RHS) is $RHS = \frac{1(1+1)}{2} = \frac{2}{2} = 1$.

Therefore the statement is true for $n = 1$

Step 2: Let assume that the statement is true for $n = k$, i.e.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

In that case, for $k + 1$, the LHS of the equality is:

$LHS = 1 + 2 + 3 + \cdots + k + (k + 1)$

$$LHS = \frac{k(k+1)}{2} + (k+1)$$ using the assumption above

$$LHS = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$LHS = \frac{k(k+1) + 2(k+1)}{2}$$

$$LHS = \frac{(k+1)(k+2)}{2}$$ which is the same statement as the assumption above, for $(k + 1)$

Therefore, if the statement is true for k, then it is true for $(k + 1)$

Step 3:

The statement is true for $n = 1$

The statement is true for $(k + 1)$ if it is true for k.

Therefore, by induction, it is true for all $n \geq 1$

Question 2: Prove by mathematical induction that for all positive integer values of n:

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$$

Step 1: When $n = 1$, the left-hand side (LHS) is $LHS = 1$, whereas the right-hand side (RHS) is $RHS = 2^1 - 1 = 1$

Therefore the statement is true for $n = 1$

Step 2: Let assume that the statement is true for $n = k$, i.e.

$$1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$$

In that case, for $k + 1$, the LHS of the equality is:

$LHS = 1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^k$

$LHS = (2^k - 1) + 2^k$ using the assumption above

$LHS = 2 \times 2^k - 1$

$LHS = 2^{k+1} - 1$

$LHS = 2^{k+1} - 1$ which is the same statement as the assumption above, for $(k + 1)$

Therefore, if the statement is true for k, then it is true for $(k+1)$

Step 3:

The statement is true for $n = 1$

The statement is true for $(k + 1)$ if it is true for k.

Therefore, by induction, it is true for all $n \geq 1$

Question 3: Prove by mathematical induction that for all positive integer values of n:

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Step 1: When $n = 1$, the left-hand side (LHS) is $LHS = 1$, whereas the right hand side (RHS) is $RHS = \frac{1(1+1)(2\times1+1)}{6} = \frac{2\times3}{6} = 1$.

Therefore the statement is true for $n = 1$

Step 2: Let assume that the statement is true for $n = k$, i.e.

$$1 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

In that case, for $k + 1$, the LHS of the equality is:

$LHS = 1 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2$

$$LHS = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$ using the assumption above

$$LHS = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$LHS = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$LHS = \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$LHS = \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$LHS = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$LHS = \frac{(k+1)(k+2)[2(k+1)+3]}{6}$$ which is the same statement as the assumption above, for $(k+1)$

Therefore, if the statement is true for k, then it is true for k+1

Step 3:

The statement is true for n = 1

The statement is true for $(k+1)$ if it is true for k

Therefore, by induction, it is true for all $n \in \mathbb{N}$

Question 4: Prove by mathematical induction that for all positive integer values of n:

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Step 1: For $n=1$, the left hand side (LHS) is $LHS=\frac{1}{1\times 3}$, whereas the right hand side (RHS) is $RHS=\frac{1}{2\times 1+1}=\frac{1}{3}$.
Therefore the statement is true for $n=1$

Step 2: Let assume that the statement is true for $n=k$, i.e.

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

In that case, for $k+1$, the LHS of the equality is:

$$LHS=\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$LHS=\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$LHS=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$ using the assumption above

$$LHS=\frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}$$

$$LHS=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$LHS=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$LHS=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$LHS=\frac{k+1}{2k+3}$$

$$LHS=\frac{k+1}{2(k+1)+1}$$ which is the same statement as the assumption above, for $k+1$

Therefore, if the statement is true for k, then it is true for $k+1$

Step 3:

The statement is true for $n=1$

The statement is true for $k+1$ if it is true for k.

Therefore, by induction, it is true for all $n\in\mathbb{N}$