RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS

Quadratic equations

The general quadratic equation is: $ax^2 + bx + c = 0,\ a \neq 0$

Dividing both sides by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ [1]

Let the roots of this equation be α and β.

Hence: $(x-\alpha)(x-\beta) = 0$

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$ [2]

i.e. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

Equating the coefficients of like powers of x in [1] and [2]:

Sum of roots: $\alpha + \beta = -\frac{b}{a}$ [3]

Product of roots: $\alpha\beta = \frac{c}{a}$ [4]

The equations [3] and [4] show the relationships between the roots α, β and the coefficients a, b, c of a quadratic equation.

Example 10

Write the quadratic equation with roots that are the squares of the roots of $2x^2 + 3x + 5 = 0$.

Solution

This can be answered without solving the original equation.

Let the roots of $2x^2 + 3x + 5 = 0$ be α and β.

Now, $a = 2$, $b = 3$, $c = 5$: $\alpha + \beta = -\frac{3}{2}$ and $\alpha\beta = \frac{5}{2}$

The new equation is of the form: $x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$

But $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$, so $\alpha^2 + \beta^2 = \left(-\frac{3}{2}\right)^2 - 2 \times \frac{5}{2} = \frac{9}{4} - 5 = -\frac{11}{4}$

and $\alpha^2\beta^2 = \frac{25}{4}$

Hence the required equation is: $x^2 + \frac{11}{4}x + \frac{25}{4} = 0$

i.e. $4x^2 + 11x + 25 = 0$

Two useful identities:

1. $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
2. $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

Cubic equations (degree 3)

The general cubic equation is: $ax^3 + bx^2 + cx + d = 0,\ a \neq 0$

Dividing both sides by a: $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$ [1]

Let the roots of this equation be α, β and γ.

Hence: $(x-\alpha)(x-\beta)(x-\gamma) = 0$

$x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma = 0$ [2]

i.e. $x^3 - (\text{sum of roots})x^2 + (\text{sum of products of each pair of roots})x - (\text{product of roots}) = 0$

Equating the coefficients of like powers of x in [1] and [2]:

Sum of roots: $\alpha + \beta + \gamma = -\frac{b}{a}$ [3]

Sum of products of each pair of roots: $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ [4]

Product of roots: $\alpha\beta\gamma = -\frac{d}{a}$ [5]

The equations [3], [4] and [5] show the relationships between the roots α, β, γ and the coefficients a, b, c, d of a cubic equation.

Note: These relationships between the roots and the coefficients are not enough to find the roots of an equation without some additional information.

Example 11

Show that $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$.

Solution

$$\begin{aligned}
\text{RHS} &= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma) \\
&= \alpha^2 + \alpha\beta + \alpha\gamma + \alpha\beta + \beta^2 + \beta\gamma + \alpha\gamma + \beta\gamma + \gamma^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) \\
&= \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma - 2(\alpha\beta + \alpha\gamma + \beta\gamma) \\
&= \alpha^2 + \beta^2 + \gamma^2 = \text{LHS}
\end{aligned}$$

Another useful identity:

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$$

Example 12

If α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x + 4 = 0$, find the value of the following.

(a) $\alpha + \beta + \gamma$ **(b)** $\alpha\beta + \alpha\gamma + \beta\gamma$ **(c)** $\alpha\beta\gamma$ **(d)** $(\alpha - 1)(\beta - 1)(\gamma - 1)$

(e) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **(f)** $\alpha^2 + \beta^2 + \gamma^2$ **(g)** $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

Solution

(a) Sum of roots $= \alpha + \beta + \gamma = -\frac{b}{a} = -2$

(b) Sum of products of pairs of roots $= \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 3$

(c) Product of roots $= \alpha\beta\gamma = -\frac{d}{a} = -4$

(d)
$$\begin{aligned}(\alpha - 1)(\beta - 1)(\gamma - 1) &= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1 \\ &= -4 - 3 - 2 - 1 \\ &= -10\end{aligned}$$

(e)
$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} \\ &= -\frac{3}{4}\end{aligned}$$

(f)
$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) \\ &= (-2)^2 - 2 \times 3 \\ &= -2\end{aligned}$$

(g)
$$\begin{aligned}\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} &= \frac{\gamma + \beta + \alpha}{\alpha\beta\gamma} \\ &= \frac{-2}{-4} \\ &= \frac{1}{2}\end{aligned}$$

Quartic equations (4th degree)

The general quartic equation is: $ax^4 + bx^3 + cx^2 + dx + e = 0, a \neq 0$

Dividing both sides by a: $x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$ [1]

Let the roots of this equation be α, β, γ and δ.

Hence: $(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) = 0$

$x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta = 0$ [2]

i.e. x^4 – (sum of roots)x^3 + (sum of products of each pair of roots)x^2
– (sum of products of each triplet of roots)x + (product of roots) = 0

Equating the coefficients of like powers of x in [1] and [2]:

Sum of roots: $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$ [3]

Sum of products of pairs of roots: $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$ [4]

Sum of products of triplets of roots: $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$ [5]

Product of roots: $\alpha\beta\gamma\delta = \frac{e}{a}$ [6]

Example 13

Solve the equation $2x^3 - 7x^2 - 12x + 45 = 0$, given that two of its roots are equal.

Solution

Let the roots be α, α, β.

Sum of roots: $2\alpha + \beta = \frac{7}{2}$ [1]

Sum of products of pairs of roots: $\alpha^2 + 2\alpha\beta = -6$ [2]

Product of roots: $\alpha^2\beta = -\frac{45}{2}$ [3]

From [1]: $\beta = \frac{7}{2} - 2\alpha$

Substitute into [2]:

$$\alpha^2 + 2\alpha\left(\frac{7}{2} - 2\right) = -6$$
$$\alpha^2 + 7\alpha - 4\alpha^2 = -6$$
$$3\alpha^2 - 7\alpha - 6 = 0$$
$$(3\alpha + 2)(\alpha - 3) = 0$$
$$\alpha = 3 \quad \textbf{or} \quad -\frac{2}{3}$$

Substitute into [1]: $\beta = -\frac{5}{2}$ **or** $\frac{29}{6}$

Substitute $\alpha = 3, \beta = -\frac{5}{2}$ into [3]: $\text{LHS} = 3^2 \times \left(-\frac{5}{2}\right) = -\frac{45}{2} = \text{RHS}$

Substitute $\alpha = -\frac{2}{3}, \beta = \frac{29}{6}$ into [3]: $\text{LHS} = \left(-\frac{2}{3}\right)^2 \times \frac{29}{6} = -\frac{58}{27} \neq \text{RHS}$

Therefore, the roots of the equation are 3, 3, –2.5.

Substitute into [3]:

$$-\frac{p}{3}\left(\frac{p^2}{9} - \left(\frac{p^2}{3} - q\right)\right) = -r$$
$$p\left(\frac{p^2}{9} - \frac{p^2}{3} + q\right) = 3r$$
$$p(p^2 - 3p^2 + 9q) = 27r$$
$$p(9q - 2p^2) = 27r$$