Example 27

A large cube of ice has an edge length of 10 cm. It melts so that its volume decreases at a constant rate of 25 cm^3 per hour. Find:

(a) the volume V at time t **(b)** the time required to completely melt the ice.

Solution

(a) Because the volume decreases at a constant rate of 25 cm^3 per hour, $\frac{dV}{dt} = -25$.

$$\therefore V = -\int 25\,dt$$
$$= -25t + C$$

When $t = 0$, $V = 10^3 = 1000$: $1000 = C$

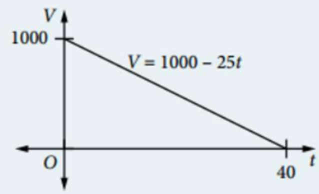
$$\therefore V = 1000 - 25t$$

(b) When $V = 0$: $0 = 1000 - 25t$

$t = 40$

The volume at any time t is given by $V = 1000 - 25t$, $0 \le t \le 40$.

Note that the domain of the volume function is restricted to values of t from 0 to 40. The volume function is a linear function and its graph has a constant gradient of -25.



Example 28

A chemical solution is being filtered into an empty beaker. The rate at which the level of the solution is rising in the beaker is given by $\frac{dh}{dt} = \frac{1}{10}\left(1 - \frac{t}{100}\right)$, where h is the depth of the solution in cm and t is the time in seconds after the solution starts flowing through the filter.

(a) What is the depth of the solution in the beaker after 40 seconds?

(b) What will be the depth of the solution in the beaker when the solution has stopped flowing through the filter?

Solution 1

(a) When $t = 0, h = 0$: $\frac{dh}{dt} = \frac{1}{10}\left(1 - \frac{t}{100}\right)$

$$h = \int \frac{1}{10}\left(1 - \frac{t}{100}\right)dt$$
$$= \frac{1}{10}\left(t - \frac{t^2}{200}\right) + C$$

When $t = 0, h = 0$: $\therefore C = 0$

Hence: $h = \frac{1}{10}\left(t - \frac{t^2}{200}\right)$

When $t = 40$: $h = \frac{1}{10}\left(40 - \frac{1600}{200}\right)$

$h = 3.2$

The depth of the solution after 40 seconds is 3.2 cm.

(b) The solution will have stopped flowing when $\frac{dh}{dt} = 0$, as at this time the depth of the solution has stopped changing.

When $\frac{dh}{dt} = 0$: $\frac{1}{10}\left(1 - \frac{t}{100}\right) = 0$

$t = 100$

Stops flowing after 100 seconds.

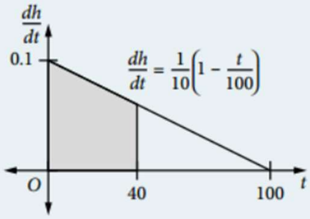
When $t = 100$: $h = \frac{1}{10}\left(100 - \frac{100^2}{200}\right) = 5$

The depth when it stops flowing is 5 cm.

Solution 2

(a) $h = \int dh = \int \frac{dh}{dt}\,dt$, so the area under $\frac{dh}{dt}$ can be used to find the value of h between appropriate limits.

Sketch $\frac{dh}{dt} = \frac{1}{10}\left(1 - \frac{t}{100}\right)$:



The shaded region is bounded by $\frac{dh}{dt} = \frac{1}{10}\left(1 - \frac{t}{100}\right)$, $\frac{dh}{dt} = 0$, $t = 0$, $t = 40$, because the depth is after 40 seconds.

$$\text{Shaded area} = \int_0^{40} \frac{1}{10}\left(t - \frac{t}{100}\right)dt$$
$$= \frac{1}{10}\left[t - \frac{t^2}{200}\right]_0^{40}$$
$$= \frac{1}{10}\left(40 - \frac{1600}{200} - 0\right)$$
$$= 3.2$$

The shaded area represents the depth of the solution after 40 seconds, so:

$$h = \int_0^{40} \frac{1}{10}\left(t - \frac{t}{100}\right)dt$$
$$= 3.2\,\text{cm}$$

(b) The graph shows that $\frac{dh}{dt} = 0$ when $t = 100$, so this is when the water has stopped flowing.

$$h = \int_0^{100} \frac{1}{10}\left(t - \frac{t}{100}\right)dt$$
$$= \frac{1}{10}\left[t - \frac{t^2}{200}\right]_0^{100}$$
$$= \frac{1}{10}\left(100 - \frac{10000}{200} - 0\right)$$
$$= 5\,\text{cm}$$

Units and Symbols

Physical quantity	Unit	Symbol
Time	s	t
Displacement	cm, m	x (or s in Physics)
Velocity	cm s^{-1}, m s^{-1}	v, $\frac{dx}{dt}$, $\dot{x}$
Acceleration	cm s^{-2}, m s^{-2}	a, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $\ddot{x}$

Note that ‘s’ is the abbreviation for second, ‘cm’ for centimetre and ‘m’ for metre.

Constant acceleration due to gravity = $9.8\,\text{m s}^{-2}$ ($\approx 10\,\text{m s}^{-2}$)

Example 29

A particle starts from rest 5 m from a fixed point O and moves in a straight line with an acceleration a m s^{-1}, where $a = 3t - 4$. Find the velocity and position of the particle at any time t.

Solution

When $t = 0, v = 0, x = 5$: $\quad a = 3t - 4$

Integrate for v: $\quad v = \int (3t - 4)\,dt$

$$v = \frac{3t^2}{2} - 4t + C_1$$

When $t = 0, v = 0$: $\quad 0 = 0 - 0 + C_1$

$$C_1 = 0$$

Hence: $\quad v = \frac{3t^2}{2} - 4t$

Integrate for x: $\quad x = \int \left(\frac{3t^2}{2} - 4t\right) dt$

$$x = \frac{t^3}{2} - 2t^2 + C_2$$

When $t = 0, x = 5$: $\quad 5 = 0 - 0 + C_2$

$$C_2 = 5$$

Hence $\quad x = \frac{t^3}{2} - 2t^2 + 5$

Example 31

A particle moves in a straight line so that at any time t seconds, its velocity v m s^{-1} is given by $v = e^{-t}$. If initially the particle is 2 metres from a fixed point O in the line, find its position x at any time t. Sketch the graph of x as a function of t.

Solution

$$v = \frac{dx}{dt} = e^{-t}$$

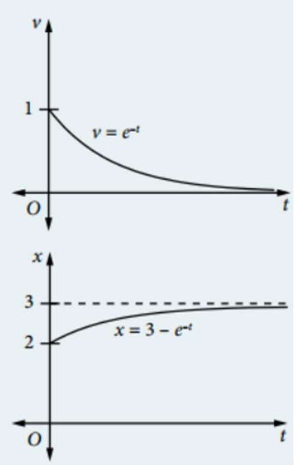
$$\therefore x = \int e^{-t}\,dt$$

$$x = -e^{-t} + C$$

When $t = 0, x = 2$: $\quad 2 = -1 + C$

$$C = 3$$

$$x = 3 - e^{-t}$$



The velocity–time graph for $v = e^{-t}$ shows that initially the particle moves with a velocity of 1 m s^{-1}, but as time goes by this velocity gets smaller and smaller so that as $t \to \infty$, $v \to 0$ from above.

The displacement–time graph shows that the particle starts at $x = 2$ and as time goes by it moves closer and closer to $x = 3$, but it never reaches it. As $t \to \infty$, $x \to 3$ from below.

Example 30

A ball is projected vertically upwards, with a velocity of $25\,\text{m s}^{-1}$, from the top of a building 30 m high. If the acceleration due to gravity is taken to be $10\,\text{m s}^{-2}$, find:

(a) the time taken for the ball to reach its highest point, and the height of this highest point

(b) how much time the ball will take to reach the ground

(c) the speed with which the ball hits the ground

(d) when the ball is 60 m above the ground.

Solution

Take the upward direction as positive. Thus when $t = 0$, $a = -10$, $v = 25$, $x = 30$.

$$a = \ddot{x} = -10$$

Integrate for v: $\quad v = \dot{x} = -\int 10\,dt$

$$v = \dot{x} = -10t + C_1$$

When $t = 0$, $v = 25$: $\quad C_1 = 25$

$$\dot{x} = 25 - 10t$$

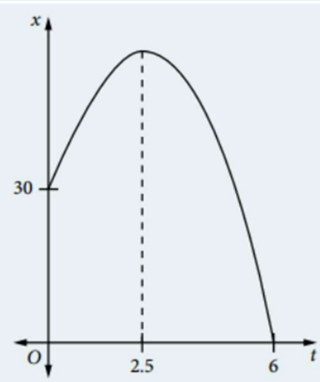
Integrate for x: $\quad x = \int (25 - 10t)\,dt$

$$x = 25t - 5t^2 + C_2$$

When $t = 0$, $x = 30$: $\quad C_2 = 30$

$$x = 30 + 25t - 5t^2$$

or $\quad x = 5(6 + 5t - t^2)$



(a) At the highest points, $\dot{x} = 0$: $\quad 0 = 25 - 10t$

$$t = 2.5$$

It takes 2.5 seconds to reach the highest point.

When $t = 2.5$: $\quad x = 5(6 + 12.5 - 6.25)$

$$x = 61.25$$

Highest point is 61.25 m above the ground.

(b) Reaches the ground when $x = 0$: $\quad 5(6 + 5t - t^2) = 0$

$$t^2 - 5t - 6 = 0$$

$$(t - 6)(t + 1) = 0$$

Because $t \geq 0$, the ball reaches the ground after 6 seconds.

(c) When $t = 6$, find $\dot{x}$: $\quad \dot{x} = 25 - 60$

$$= -35$$

The ball is falling when it hits the ground, hence the negative velocity, and it strikes the ground with a speed of $35\,\text{m s}^{-1}$.

(d) When $x = 60$: $\quad 60 = 5(6 + 5t - t^2)$

$$12 = 6 + 5t - t^2$$

$$t^2 - 5t + 6 = 0$$

$$(t - 2)(t - 3) = 0$$

$$t = 2 \text{ or } 3$$

The ball is 60 m above the ground at 2 seconds (on the way up) and again at 3 seconds (on the way down).