THE SECOND DERIVATIVE AND CONCAVITY

Differentiating a function once to obtain $f'(x)$ or $\frac{dy}{dx}$ gives you the first derivative of the original function.

Differentiating the first derivative gives you $f''(x)$ or $\frac{d^2y}{dx^2}$, which is called the **second derivative** of the original function. The second derivative is the rate of change of the first derivative (i.e. of the gradient function): $\frac{d}{dx}\left(f'(x)\right)$.

Differentiating again will give the third derivative, and so on. The differentiation process may be continued for as long as a derivative exists.

Notation

Several different notations can be used for derivatives. If $y = f(x)$, then:

- the first derivative can be written $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}\left(f(x)\right)$ or y'
- the second derivative can be written $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, $\frac{d^2y}{dx^2}$, $f''(x)$, $\frac{d}{dx}\left(f'(x)\right)$ or y''.

Example 7

Find the second derivative of each function.

(a) $y = 4x^3 - 2x^2 + 3x + 7$ **(b)** $f(x) = (2x+1)^5$ **(c)** $y = \frac{x^2}{x+1}$

Solution

(a) $y = 4x^3 - 2x^2 + 3x + 7$:

$$\frac{dy}{dx} = 12x^2 - 4x + 3$$

$$\frac{d^2y}{dx^2} = 24x - 4$$

(b) $f(x) = (2x+1)^5$:

$$f'(x) = 5(2x+1)^4 \times 2 = 10(2x+1)^4$$

$$f''(x) = 10 \times 4(2x+1)^3 \times 2 = 80(2x+1)^3$$

(c) $y = \frac{x^2}{x+1}$:

$$y' = \frac{2x(x+1) - x^2 \times 1}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$$

$$y'' = \frac{(2x+2)(x+1)^2 - \left(x^2+2x\right) \times 2(x+1)}{(x+1)^4}$$

$$= \frac{2(x+1)^3 - 2(x+1)\left(x^2+2x\right)}{(x+1)^4}$$

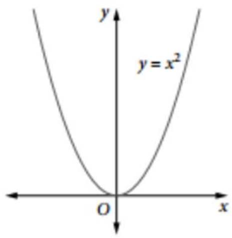
$$= \frac{2(x+1)\left((x+1)^2 - \left(x^2+2x\right)\right)}{(x+1)^4}$$

$$= \frac{2\left(x^2 + 2x + 1 - x^2 - 2x\right)}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3}$$

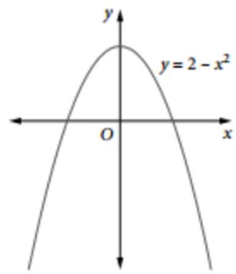
Concavity

The **concavity** of a function describes the general curvature of a graph of a non-linear function. Graphs can be 'concave upwards' and 'concave downwards':



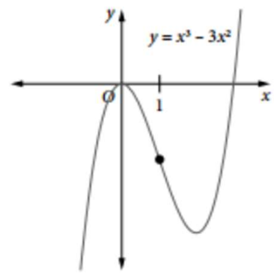
The function is concave up.

Note: This has a minimum turning point.



The function is concave down.

Note: This has a maximum turning point.



The left-hand part is concave down.

The right-hand part is concave up.

Concavity changes at $x = 1$.

The second derivative is the rate at which the first derivative is changing.
This gives information about the concavity of a function.

THE SECOND DERIVATIVE AND CONCAVITY

In each diagram below, a series of tangents have been drawn.

1 The tangents have a positive gradient and the gradient is increasing from left to right.

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} > 0$.

The rate at which the gradient is increasing $= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

Because the gradient is increasing, $\frac{d^2y}{dx^2} > 0$: the curve is concave up.

2 The tangents have a negative gradient and the gradient is increasing (becoming less negative) from left to right.

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} < 0$.

Because the gradient is increasing, $\frac{d^2y}{dx^2} > 0$: the curve is concave up.

3 The tangents have a negative gradient and the gradient is decreasing (becoming more negative) from left to right.

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} < 0$.

Because the gradient is decreasing, $\frac{d^2y}{dx^2} < 0$: the curve is concave down.

4 The tangents have a positive gradient and the gradient is decreasing from left to right.

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} > 0$.

Because the gradient is decreasing, $\frac{d^2y}{dx^2} < 0$: the curve is concave down.

5 The gradient is positive.

Initially the gradient is decreasing, $\frac{d^2y}{dx^2} < 0$, but then it starts increasing, $\frac{d^2y}{dx^2} > 0$.

This means that at some point $\frac{d^2y}{dx^2} = 0$. This is also where the concavity changes from concave down to concave up, so this point is called a **point of inflection**.

6 The gradient is negative.

Initially the gradient is increasing, $\frac{d^2y}{dx^2} > 0$, but then it starts decreasing, $\frac{d^2y}{dx^2} < 0$.

This means that at some point $\frac{d^2y}{dx^2} = 0$. This is also where the concavity changes from concave down to concave up, so this is a point of inflection.

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The sign of the second derivative

- If $\frac{d^2y}{dx^2} > 0$ on an interval then the curve is concave upwards on that interval.
- If $\frac{d^2y}{dx^2} < 0$ on an interval then the curve is concave downwards on that interval.
- If $\frac{d^2y}{dx^2} = 0$ at a point on the curve and the concavity changes at this point, then the point is called a **point of inflection**.

Example 8

For what values of x is the curve given by $y = x^3 - 3x^2 + 6x + 3$:

(a) concave up **(b)** concave down?

(c) Find the coordinates of the point of inflection. **(d)** Sketch the curve.

Solution

Find $\frac{dy}{dx}$: $\frac{dy}{dx} = 3x^2 - 6x + 6$

Find $\frac{d^2y}{dx^2}$: $\frac{d^2y}{dx^2} = 6x - 6$

(a) Concave up, $\frac{d^2y}{dx^2} > 0$: $6x - 6 > 0$, $x > 1$

The curve is concave up for $x > 1$.

(b) Concave down, $\frac{d^2y}{dx^2} < 0$: $x < 1$

The curve is concave down for $x < 1$.

(c) Inflection point, $\frac{d^2y}{dx^2} = 0$: $x = 1$

$x < 1$, curve is concave down
$x > 1$, curve is concave up
$\therefore$ concavity changes at $x = 1$
$x = 1, y = 1 - 3 + 6 + 3 = 7$
$\therefore$ point of inflection is $(1, 7)$

(d)