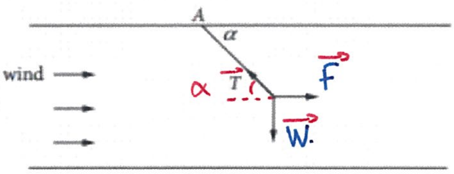
MATHEMATICAL REPRESENTATION OF MOTION IN PHYSICAL TERMS

1 A 1 kg object in a wind tunnel is suspended from a point A on the ceiling by a light rope. A wind equivalent to a horizontal force of 4.9 N is directed down the tunnel. This causes the rope and the object to move from their position vertically below A to be inclined at an angle α to the ceiling. The object remains in this position while the wind is blowing.



The diagram shows the three forces (tension T newtons in the rope, the wind force and the weight force) acting on the object. Resolve the forces into horizontal and vertical components. Noting that the object is stationary, use Newton's first law of motion to find the values of α and T (Use $g = 9.8$ m s^{-2}). The correct equations are:

A $\alpha = \tan^{-1} 2$ and $T = \dfrac{2g}{\sqrt{5}}$

B $\alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$ and $T = \dfrac{2g}{\sqrt{5}}$

$\boxed{\textbf{C}}$ $\alpha = \tan^{-1} 2$ and $T = \dfrac{\sqrt{5}\,g}{2}$

D $\alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$ and $T = \dfrac{\sqrt{5}\,g}{2}$

$$\begin{cases} T\cos\alpha = F = 4.9 \\ T\sin\alpha = W = mg = 1 \times 9.8 = 9.8 = g \end{cases}$$

So $\tan\alpha = \dfrac{9.8}{4.9} = 2$

and $T^2\cos^2\alpha = 4.9^2 = 24.01$

$T^2\sin^2\alpha = 9.8^2 = 96.04$

$\therefore T^2 = 24.01 + 96.04 = 120.05$

so $T = \sqrt{120.05} = \sqrt{1.25 \times g^2} = g\sqrt{1.25} = g\sqrt{\dfrac{5}{4}} = \dfrac{g\sqrt{5}}{2}$

So $\tan\alpha = 2$, i.e. $\alpha = \tan^{-1} 2$ and $T = \dfrac{g\sqrt{5}}{2}$

Response $\boxed{C}$

2 A particle of mass 1 kg moves in a straight line such that at time t seconds its displacement from a fixed origin is x metres and its velocity is v m s^{-1}. If the resultant force is $3 + 2t$ (in newtons), find its displacement in terms of t given that $v = 1$ and $x = 2$ when $t = 0$.

$F = ma$ so as $m = 1$ $3 + 2t = a = \ddot{x}(t)$

So $\dot{x}(t) = t^2 + 3t + C$ But at $t = 0$ $v = 1$ so $C = 1$

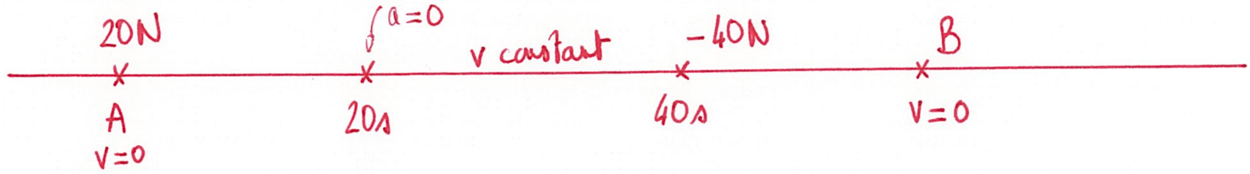
$\dot{x}(t) = t^2 + 3t + 1$

So $x(t) = \dfrac{t^3}{3} + \dfrac{3t^2}{2} + t + K$ But at $t = 0$ $x = 2$ so $K = 2$

So $x(t) = \dfrac{t^3}{3} + \dfrac{3t^2}{2} + t + 2$

4 A particle of mass 10 kg starts from rest at a point A and moves in a straight line under the action of a force that decreases uniformly from 20 N to zero in 20 seconds. The particle then travels with constant velocity for a further 20 seconds. After this, the particle moves under the action of a retarding force of 40 N until it comes to rest at point B.

(a) Express the force that applies for the first 20 seconds as a function of t.
(b) Find the velocity of the particle at $t = 20$ seconds.
(c) Find the time taken for the retarding force to stop the motion of the particle.
(d) Find the maximum speed attained during the motion.
(e) Find the total distance travelled during the motion.
(f) Find the average speed during the motion.



a) $F = 20 - t$

b) $F = ma = m\ddot{x}(t)$ so $\ddot{x}(t) = \dfrac{F}{m} = \dfrac{20-t}{m}$

So $\dot{x}(t) = \int \ddot{x}(t)\,dt = \int \dfrac{20-t}{m}\,dt = \dfrac{20t}{m} - \dfrac{t^2}{2m} + C$

But at $t=0$, $\dot{x}(0) = 0$, so $C = 0$ $\therefore$ $\dot{x}(t) = \dfrac{20t}{m} - \dfrac{t^2}{2m}$

At $t = 20\,s$ $\quad \dot{x}(20) = \dfrac{20 \times 20}{10} - \dfrac{20^2}{20} = 20\ m\,s^{-1}$

c) When the retarding force is applied, $-40 = m\ddot{x}(t)$

So $\ddot{x}(t) = -\dfrac{40}{m} = -4$ $\quad \therefore$ $\dot{x}(t) = -4t + C$

But at $t = 40\,s$, the speed is the same than at $t = 20\,s$, so

$\dot{x}(40) = 20 = -4 \times 40 + C$ $\quad$ so $C = 180$

Hence $\dot{x}(t) = -4t + 180$

$\dot{x}(t) = 0$ when $-4t + 180 = 0$ $\Leftrightarrow$ $t = \dfrac{180}{4} = 45\,s.$

So it takes 5 s for the particle to stop when the retarding force is applied.

d) Speed is at its maximum between $t=20s$ and $t=40s$, and it's $v = 20\,ms^{-1}$

e)* Between $t=0$ to $t=20s$ $\quad \dot{x}(t) = 2t - t^2/20$

So $x(t) = t^2 - \frac{t^3}{60} + C$ $\quad$ But $x(0)=0$, so $C=0$

$x(t) = t^2 - \frac{t^3}{60}$ $\quad$ so at $t=20s$ $\quad x(20) = 20^2 - \frac{20^3}{60} = 266\frac{2}{3}$

* Between $t=20s$ and $t=40s$, the speed is constant ($20\,ms^{-1}$)

So the distance travelled over these 20s at $20\,ms^{-1}$ is 400 m

* After $t=40s$, until it stops at $t=45s$, $\dot{x}(t) = -4t + 180$

so $x(t) = -2t^2 + 180t + K$

At $t=40$, $\quad x(40) = 266\frac{2}{3} + 400 = 600\frac{2}{3} = -2\times 40^2 + 180\times 40 + K$

so $K = -\frac{10{,}000}{3}$

So $x(t) = -2t^2 + 180t - \frac{10{,}000}{3}$

at $t=45s$ $\quad x(45) = -2\times 45^2 + 180\times 45 - \frac{10{,}000}{3}$

$x(45) = 716\frac{2}{3}$ m which is the total distance travelled.

f) Average speed $= \frac{716\ 2/3}{45} = 15\frac{25}{27}\,ms^{-1} \simeq 15.92\,ms^{-1}$

5 A particle of mass 10 kg moves under the action of a force so that its velocity v m s^{-1} is given by $v=\sqrt{x^2-6x+5}$. Find the force F in terms of the displacement x.

$F = m\ddot{x}(t) = 10\,\ddot{x}(t)$

$\dot{x}(t) = \sqrt{x^2-6x+5}$

So $\ddot{x}(t) = \frac{d}{dt}\dot{x}(t) = \frac{d}{dt}\left[\sqrt{x^2-6x+5}\right] = \frac{d}{dx}\left[\sqrt{x^2-6x+5}\right] \times \frac{dx}{dt}$

$\ddot{x}(t) = \frac{1}{2} \times \frac{(2x-6)}{\sqrt{x^2-6x+5}} \times \dot{x}(t) = \frac{x-3}{\sqrt{x^2-6x+5}} \times \sqrt{x^2-6x+5}$

So $\ddot{x}(t) = x-3$

and therefore : $F = 10 \times (x-3) = 10x - 30$

6 A particle of mass m moves so that its velocity v is given by $v=f(x)$. The resultant force that causes this motion is given by:

A $mf'(x)$ **B** $mxf'(x)$ **C** $mf'\left(\frac{1}{2}x^2\right)$ **D** $mf(x)f'(x)$

$\dot{x}(t) = f(x)$ $F = m\ddot{x}(t)$

$\ddot{x}(t) = \frac{d}{dt}\dot{x}(t) = \frac{d}{dx}\left[\dot{x}(t)\right] \times \frac{dx}{dt} = \frac{d}{dx}\left(f(x)\right) \times \dot{x}(t)$

So $\ddot{x}(t) = f'(x) \times f(x)$

$\therefore F = m\, f(x) \times f'(x)$

9 At time t a particle of mass m is moving in a straight line under the action of a force given by $F = \frac{m(3-5x)}{x^3}$. The particle starts from rest at $x = \frac{1}{3}$.

(a) Find its velocity in terms of x.

(b) At what other point, if any, does the particle come to rest?

a) $F = m\ddot{x}(t)$ so $\ddot{x}(t) = \frac{F}{m} = \frac{3-5x}{x^3} = \frac{3}{x^3} - \frac{5}{x^2}$

But $\ddot{x}(t) = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}[\dot{x}(t)]^2\right)$

So $\frac{d}{dx}\left[\frac{1}{2}v^2\right] = \frac{3}{x^3} - \frac{5}{x^2}$

$\therefore \frac{1}{2}v^2 = \int\left[\frac{3}{x^3} - \frac{5}{x^2}\right]dx$

$\frac{1}{2}v^2 = \int(3x^{-3} - 5x^{-2})dx = 3\frac{x^{-2}}{(-2)} - 5\frac{x^{-1}}{(-1)} + C$

$\therefore \frac{1}{2}v^2 = -\frac{3}{2x^2} + \frac{5}{x} + C$

at $x = 1/3$ $v = 0$, so $0 = -\frac{3}{2\times\left(\frac{1}{3}\right)^2} + \frac{5}{\left(\frac{1}{3}\right)} + C$

so $C = -15 + \frac{27}{2} = -\frac{3}{2}$

$\therefore v^2 = -\frac{3}{x^2} + \frac{10}{x} - 3 = \frac{-3x^2+10x-3}{x^2}$

therefore $v = \pm\frac{\sqrt{-3x^2+10x-3}}{x}$

b) $v=0$ when $-3x^2+10x-3=0$ $\Delta = 100 - 4\times3\times3 = 64$

$x_1 = \frac{-10+8}{(-6)} = \frac{-2}{-6} = \frac{1}{3}$ (we knew this one)

$x_2 = \frac{-10-8}{-6} = +18/6 = +3$ so also at $x = 3$

10 A particle of unit mass is acted on by a force $F = v^2 \log_e v$, where $v \text{ m s}^{-1}$ is the velocity of the particle. The motion starts from O with a velocity of $e \text{ m s}^{-1}$. Find the displacement when the velocity is $e^2 \text{ m s}^{-1}$.

$F = m\ddot{x}(t) = \ddot{x}(t)$ as $m = 1$ so $\ddot{x}(t) = v^2 \ln v$

We require x in term of v, so $v \dfrac{dv}{dx} = v^2 \ln v$

$\iff \dfrac{dv}{dx} = v \ln v$

$\iff dx = \dfrac{dv}{v \ln v}$

So, integrating both sides: $x = \displaystyle\int \frac{1}{v \ln v} \, dv$

$x = \displaystyle\int \frac{1/v}{\ln v} \, dv$ Change of variable $u = \ln v$

so $\dfrac{du}{dv} = \dfrac{1}{v} \Rightarrow \dfrac{dv}{v} = du$

$x = \displaystyle\int \frac{du}{u} = \ln |u| + C$

$x = \ln |\ln v| + C$

At $x = 0$, $v = e$, so $0 = \ln |\ln e| + C = \underbrace{\ln 1}_{=0} + C$

So $C = 0$

$x = \ln |\ln v| + C$

When $v = e^2$, $x = \ln |\ln e^2| = \ln |2 \ln e| = \ln 2$ m

11 The acceleration of an object moving towards a planet under gravitational attraction varies inversely as the square of the distance from the centre of the planet (i.e. $\ddot{x} = -\frac{k}{x^2}$ where x is the displacement from the centre of the planet and k is a constant). Show that if the object starts from rest at a distance a from the centre of the planet, its speed at distance x from the centre of the planet is: $\sqrt{\frac{2k(a-x)}{ax}}$

$\ddot{x} = -\frac{k}{x^2}$ we want $\dot{x}$ as a function of x

As $\ddot{x} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ then $\frac{d}{dx}\left[\frac{1}{2}v^2\right] = -\frac{k}{x^2}$

So $\frac{1}{2}v^2 = \int -k\frac{dx}{x^2} = -k\frac{x^{-1}}{(-1)} + C = \frac{k}{x} + C$

At $x = a$, $v = 0$ so $C = -\frac{k}{a}$ so $\frac{1}{2}v^2 = \frac{k}{x} - \frac{k}{a}$

So $v^2 = 2k\left[\frac{a-x}{ax}\right]$

$\therefore \quad v = \sqrt{\frac{2k(a-x)}{ax}}$

12 A particle of unit mass starts from rest with displacement b (where $b > 0$) and is attracted towards the origin O with an acceleration of magnitude $\frac{k}{x^2}$ where x is the displacement from the origin and k is a positive constant.

(a) Explain why $\ddot{x} = -\frac{k}{x^2}$.

(b) Show that the velocity v is given by: $v^2 = 2k\left(\frac{1}{x} - \frac{1}{b}\right)$

(c) Use the substitution $x = b\cos^2\theta$ to show that: $\int\sqrt{\frac{x}{b-x}}\,dx = -\sqrt{bx - x^2} - \frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right) + C$

(d) Hence show that the time required for the particle to reach the origin is: $\pi\left(\frac{b^3}{8k}\right)^{\frac{1}{2}}$

a) $b > 0$, so the particle is to the right of the origin, and it moves towards the origin. So the acceleration must be negative so that it moves towards the origin, and not away from it.

So $\ddot{x} = -k/x^2$

b) $\ddot{x} = \frac{d}{dx}\left[\frac{1}{2}v^2\right]$ so $\frac{d}{dx}\left[\frac{1}{2}v^2\right] = -\frac{k}{x^2}$

So $\frac{1}{2}v^2 = -k\int\frac{dx}{x^2} = -k\frac{x^{-1}}{(-1)} + C = \frac{k}{x} + C$

At $x = b$, $v = 0$, so $\frac{k}{b} = -C$ so $C = -\frac{k}{b}$

So $\frac{1}{2}v^2 = k\left[\frac{1}{x} - \frac{1}{b}\right] \Rightarrow v^2 = 2k\left(\frac{1}{x} - \frac{1}{b}\right)$

c) $\int\sqrt{\frac{x}{b-x}}\,dx = I$ let $x = b\cos^2\theta$

so $\frac{dx}{d\theta} = 2b\cos\theta \times (-\sin\theta)$

so $dx = -2b\sin\theta\cos\theta\,d\theta$

$\therefore I = \int\sqrt{\frac{b\cos^2\theta}{b - b\cos^2\theta}} \times \left[-2b\sin\theta\cos\theta\right]d\theta = \int\frac{\cos\theta}{\sin\theta}\left(-2b\sin\theta\cos\theta\right)d\theta$

$= -b\int 2\cos^2\theta\,d\theta$

$\rightarrow$

So $I = -b\int 2\cos^2\theta \, d\theta = -b\int(\cos 2\theta + 1)\,d\theta = -b\left[\frac{\sin 2\theta}{2} + \theta\right] + C$

$\underline{I} = -b\sin\theta\cos\theta - b\theta + C$

$x = b\cos^2\theta$ so $\cos\theta = \sqrt{\frac{x}{b}}$

$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{x}{b}$ so $\sin\theta = \sqrt{\frac{b-x}{b}}$

From $x = b\cos^2\theta = \frac{b}{2}\left[\cos 2\theta + 1\right]$

so $\frac{2x}{b} - 1 = \cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1}\left(\frac{2x-b}{b}\right)$

$\therefore \int\sqrt{\frac{x}{b-x}}\,dx = -b\sqrt{\frac{b-x}{b}}\sqrt{\frac{x}{b}} - b \times \frac{1}{2}\cos^{-1}\left(\frac{2x-b}{b}\right) + C$

OR $\int\sqrt{\frac{x}{b-x}}\,dx = -\sqrt{bx - x^2} - \frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right) + C$

d) $\frac{dx}{dt} = v = \sqrt{2k\left(\frac{b-x}{bx}\right)} = \sqrt{\frac{2k}{b}}\sqrt{\frac{b-x}{x}}$ so $\frac{dt}{dx} = \sqrt{\frac{b}{2k}}\sqrt{\frac{x}{b-x}}$ that we integrate

$t_{origin} - \underbrace{t_b}_{=0} = \sqrt{\frac{b}{2k}}\left[-\sqrt{bx-x^2} - \frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right)\right]_b$

$t_{origin} = \sqrt{\frac{b}{2k}}\left[\sqrt{bx-x^2} + \frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right)\right]_0^b$

$\text{—} = \sqrt{\frac{b}{2k}}\left\{\left[\sqrt{b^2-b^2} + \frac{b}{2}\cos^{-1}(1)\right] - \left[\sqrt{0} + \frac{b}{2}\cos^{-1}(-1)\right]\right\}$

$\text{—} = \sqrt{\frac{b}{2k}}\left\{\frac{b}{2} \times 0 - \frac{b}{2} \times (-\pi)\right\}$

$\text{—} = \pi\frac{b^{3/2}}{2^{3/2}k^{1/2}} = \pi\left(\frac{b^3}{8k}\right)^{1/2}$

15 A particle moves in a straight line with simple harmonic motion. Its speed at distances x_1, x_2 from the centre of its motion are v_1, v_2 respectively. Show that:

(a) the period of the motion is $2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$ **(b)** the amplitude is $\sqrt{\dfrac{v_1^2x_2^2 - v_2^2x_1^2}{v_1^2 - v_2^2}}$

a) $x(t) = a\cos(nt+\alpha)$. so $a\cos(nt+\alpha) = x(t)$

$\dot{x}(t) = -an\sin(nt+\alpha)$ so $a\sin(nt+\alpha) = -\dfrac{\dot{x}(t)}{n}$

So $a^2\cos^2(nt+\alpha) + a^2\sin^2(nt+\alpha) = [x(t)]^2 + \dfrac{[\dot{x}(t)]^2}{n^2}$

So $a^2 = x^2 + \dfrac{v^2}{n^2}$

So $\dfrac{v^2}{n^2} = a^2 - x^2$ so $\dfrac{n^2}{v^2} = \dfrac{1}{a^2-x^2}$ $n = \dfrac{v}{\sqrt{a^2-x^2}}$

or $x_1^2 + \dfrac{v_1^2}{n^2} = x_2^2 + \dfrac{v_2^2}{n^2}$ so $\dfrac{v_1^2 - v_2^2}{n^2} = x_2^2 - x_1^2$

so $n^2 = \dfrac{v_1^2 - v_2^2}{x_2^2 - x_1^2}$ so $n = \sqrt{\dfrac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$

Period is $\dfrac{2\pi}{n} = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

b) $a^2 = x_1^2 + \dfrac{v_1^2}{\left[\dfrac{v_1^2 - v_2^2}{x_2^2 - x_1^2}\right]} = x_1^2 + \dfrac{v_1^2[x_2^2 - x_1^2]}{v_1^2 - v_2^2}$

$a^2 = \dfrac{\cancel{x_1^2v_1^2} - x_1^2v_2^2 + v_1^2x_2^2 - \cancel{v_1^2x_1^2}}{v_1^2 - v_2^2}$

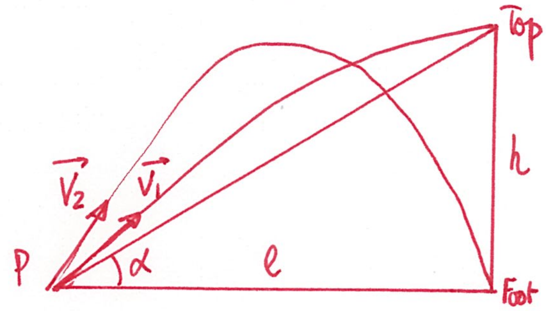
So $a^2 = \dfrac{v_1^2x_2^2 - v_2^2x_1^2}{v_1^2 - v_2^2}$ so $a = \sqrt{\dfrac{v_1^2x_2^2 - v_2^2x_1^2}{v_1^2 - v_2^2}}$

17 A vertical pole subtends an angle α at a point P in the same horizontal plane as the foot of the pole. Two particles are projected at the same instant from P in directions that make angles α_1 and α_2 with the horizontal, with initial speeds v_1 and v_2, so that the first particle hits the top of the pole at the same instant that the second particle hits the bottom.

(a) Show that: $v_1 \cos\alpha_1 = v_2 \cos\alpha_2$

(b) Show that the time of flight is: $\dfrac{2v_2 \sin\alpha_2}{g}$

(c) Hence prove that: $\tan\alpha = \tan\alpha_1 - \tan\alpha_2$



a) The horizontal speed of the 1st particle is $V_1 \cos\alpha_1$, whereas the horizontal speed of the second particle is $V_2 \cos\alpha_2$. The particles reach the same x at the same time, therefore these 2 horizontal speeds must be equal, i.e. $V_1 \cos\alpha_1 = V_2 \cos\alpha_2$.

b) $\ddot{x}_y = -g$ so $\dot{x}_y = -gt + C$

So $V_y = -gt + C$. At $t=0$ $V_y = V_2 \sin\alpha_2$

so $V_y = -gt + V_2 \sin\alpha_2$

$V_y = 0$ when $gt = V_2 \sin\alpha_2$ so $t = \dfrac{V_2 \sin\alpha_2}{g}$

So it takes $\left[\dfrac{V_2 \sin\alpha_2}{g}\right]$ for the particle to reach the top of the parabola.

And as long to come down. So in total, $\dfrac{2V_2 \sin\alpha_2}{g}$

c) $\tan\alpha = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{h}{\ell}$

$\ell = \text{speed} \times \text{time} = V_1 \cos\alpha_1 \times \dfrac{2V_2 \sin\alpha_2}{g}$

By summing up the forces on particle 1 : $-mg = m\ddot{x}_1$

so $\ddot{x}_1 = -g$.

on the y-axis $V_{1y} = -gt + C$ (by integrating from above)

At $t=0$ $V_{1y}(0) = V_1 \sin\alpha_1$ so $C = V_1 \sin\alpha_1$

$$V_{1y} = -gt + V_1 \sin\alpha_1$$

Integrating again, we obtain $y_1 = -\frac{1}{2}gt^2 + V_1 \sin\alpha_1 t + K$

At $t=0$ $y_1 = 0$ so $K=0$: $y_1 = -\frac{1}{2}gt^2 + V_1 \sin\alpha_1 t$

At $t = \frac{2V_2 \sin\alpha_2}{g}$, the particle is at the top of the pole,

therefore $h = -\frac{1}{2}g\left[\frac{2V_2 \sin\alpha_2}{g}\right]^2 + V_1 \sin\alpha_1 \left[\frac{2V_2 \sin\alpha_2}{g}\right]$

$$h = \frac{2V_2 \sin\alpha_2}{g}\left[-V_2 \sin\alpha_2 + V_1 \sin\alpha_1\right].$$

$$\therefore \tan\alpha = \frac{\frac{2V_2 \sin\alpha_2}{g}\left(V_1 \sin\alpha_1 - V_2 \sin\alpha_2\right)}{\frac{2V_2 \sin\alpha_2}{g} \times V_1 \cos\alpha_1}$$

$$\tan\alpha = \frac{V_1 \sin\alpha_1 - V_2 \sin\alpha_2}{V_1 \cos\alpha_1}$$

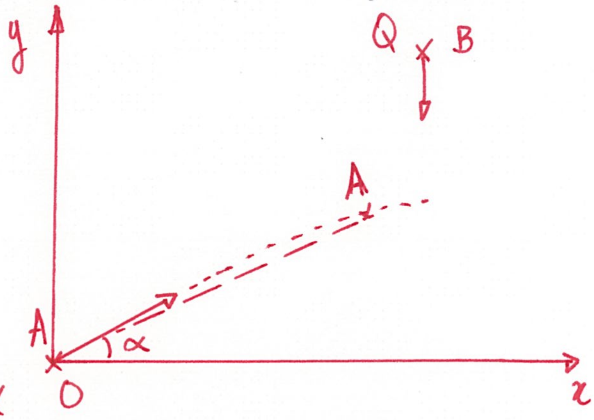
$$\tan\alpha = \frac{V_1 \sin\alpha_1}{V_1 \cos\alpha_1} - \frac{V_2 \sin\alpha_2}{V_1 \cos\alpha_1}$$ but $V_1 \cos\alpha_1 = V_2 \cos\alpha_2$

$$\tan\alpha = \tan\alpha_1 - \frac{V_2 \sin\alpha_2}{V_2 \cos\alpha_2} = \tan\alpha_1 - \tan\alpha_2$$

18 A projectile A is projected from O with speed u at an angle α above the horizontal. A package B is parachuting vertically downwards at a constant speed equal to $-u\sin\alpha$. At the moment when A is projected, the package B is at the point $Q\left(\frac{u^2}{g}\sin 2\alpha, \frac{2u^2}{g}\sin^2\alpha\right)$.

(a) Find the coordinates of A and B at time t after A is projected.

(b) Show that a searchlight, which is located at O and which is moved so that its beam is continually directed at A, will always have B in its beam.



a) $x_A = u\cos\alpha\, t$

$\ddot{y}_A = -g$ so $\dot{y}_A = -gt + C$

at $t=0$ $\dot{y}_A(0) = u\sin\alpha$

so $C = u\sin\alpha$ $\therefore$ $\dot{y}_A = -gt + u\sin\alpha$

so $y_A = -\frac{1}{2}gt^2 + u\sin\alpha\, t + K$ $\quad$ $K=0$ as $y_A(0)=0$

So $y_A = -\frac{1}{2}gt^2 + u\sin\alpha\, t$

$A\left(u\cos\alpha\, t, -\frac{1}{2}gt^2 + u\sin\alpha\, t\right).$

For B: $x_B = \frac{u^2}{g}\sin 2\alpha$ and $\dot{y}_B(t) = -u\sin\alpha$

so $y_B(t) = \frac{2u^2}{g}\sin^2\alpha - u\sin\alpha\, t$

$B\left(\frac{u^2}{g}\sin 2\alpha, \frac{2u^2}{g}\sin^2\alpha - u\sin\alpha\, t\right)$

b) $\vec{OA} = \left[u\cos\alpha\, t\right]\vec{i} + \left[u\sin\alpha\, t - \frac{1}{2}gt^2\right]\vec{j}$

$\vec{OB} = \left[\frac{u^2}{g}\sin 2\alpha\right]\vec{i} + \left[\frac{2u^2}{g}\sin^2\alpha - u\sin\alpha\, t\right]\vec{j}$

The gradient of $\overrightarrow{OA}$ is $\dfrac{u \sin\alpha t - \frac{1}{2} g t^2}{u \cos\alpha t} = \dfrac{2u\sin\alpha - gt}{2u\cos\alpha}$

whereas the gradient of $\overrightarrow{OB}$ is:

$$\frac{\frac{2u^2}{g}\sin^2\alpha - u\sin\alpha t}{\frac{u^2}{g}\sin 2\alpha} = \frac{2u^2\sin^2\alpha - gu\sin\alpha t}{2u^2\sin\alpha\cos\alpha}$$

$$= \frac{2u\sin\alpha - gt}{2u\cos\alpha}.$$

So the gradient of $\overrightarrow{OA}$ is the same than it of $\overrightarrow{OB}$.

Therefore a searchlight located at O and pointing at A will also be pointing at B.

19 Two stones are thrown simultaneously from the same point in the same direction and with the same non-zero angle of projection α (upward inclination to the horizontal), but with different velocities u, v metres per second ($u < v$).

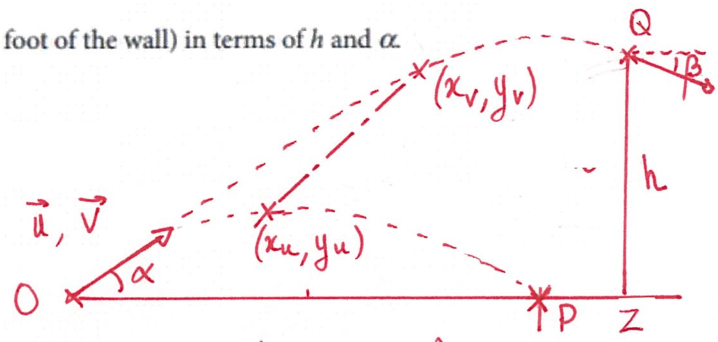
The slower stone hits the ground at a point P on the same level as the point of projection. At that instant, the faster stone just clears a wall ZQ of height h metres above the level of projection as its (downward) path makes an angle β with the horizontal.

(a) Show that while both stones are in flight, the line joining them has an inclination to the horizontal which is independent of time.

(b) Hence express the horizontal distance from P to Z (the foot of the wall) in terms of h and α.

(c) Show that $v(\tan\alpha + \tan\beta) = 2u\tan\alpha$.

(d) Deduce that if $\beta = \frac{1}{2}\alpha$ then $u < \frac{3}{4}v$.



a) The line joining them has for gradient $\left(\frac{y_v - y_u}{x_v - x_u}\right)$

$x_u = u\cos\alpha\ t$ and $x_v = v\cos\alpha\ t$

For both stones $\ddot{x}_u = \ddot{x}_v = -g$

so $\dot{y}_u = -gt + C_{uy}$ (vertical component)

at $t=0$ $\dot{y}_u(0) = u\sin\alpha$ so $C_{uy} = u\sin\alpha$

$\dot{y}_u = -gt + u\sin\alpha$

$y_u = -\frac{1}{2}gt^2 + u\sin\alpha\ t + K$ but $K=0$ as $y_u(0)=0$

$$y_u = -\frac{1}{2}gt^2 + u\sin\alpha\ t$$

Likewise $$y_v = -\frac{1}{2}gt^2 + v\sin\alpha\ t$$

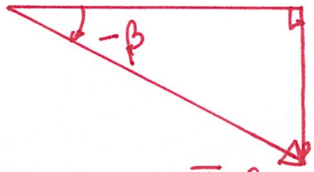
So: $$\frac{y_v - y_u}{x_v - x_u} = \frac{\left(-\frac{1}{2}gt^2 + v\sin\alpha\ t\right) - \left(-\frac{1}{2}gt^2 + u\sin\alpha\ t\right)}{v\cos\alpha\ t - u\cos\alpha\ t}$$

$$= \frac{v\sin\alpha - u\sin\alpha}{v\cos\alpha - u\cos\alpha} = \frac{\sin\alpha\,(v-u)}{\cos\alpha\,(v-u)} = \tan\alpha$$

which is constant indeed.

b) The line joining the stones has a constant incline,

$$\therefore \tan\alpha = \frac{h}{PZ} \Rightarrow PZ = \frac{h}{\tan\alpha} = h \operatorname{cotan}\alpha$$



c) $$\tan(-\beta) = \frac{\dot{y}_v}{\dot{x}_v} = \frac{-gt + v\sin\alpha}{v\cos\alpha}$$

We need to find the time t when the slower particle hits the ground; that occurs when $y_u = 0$, i.e. $-\frac{1}{2}gt^2 + u\sin\alpha\, t = 0$

i.e $-\frac{1}{2}gt + u\sin\alpha = 0$ so at $t = \frac{2u\sin\alpha}{g}$.

So $$\tan(-\beta) = \frac{-2u\sin\alpha + v\sin\alpha}{v\cos\alpha}$$

$$\tan(-\beta) = \left[-\frac{2u}{v} + 1\right]\tan\alpha.$$

$$v\tan(-\beta) = (-2u + v)\tan\alpha \Leftrightarrow -v\tan\beta - v\tan\alpha = -2u\tan\alpha$$

$$\Leftrightarrow v\left[\tan\alpha + \tan\beta\right] = 2u\tan\alpha$$

d) if $\beta = \frac{1}{2}\alpha$ then $LHS = v\left[\tan\alpha + \tan\left(\frac{\alpha}{2}\right)\right] = v\left[\frac{2\tan(\alpha/2)}{1-\tan^2(\alpha/2)} + \tan\left(\frac{\alpha}{2}\right)\right]$

$$LHS = v\left[\frac{2t}{1-t^2} + t\right] = v\left[\frac{2t + t - t^3}{1-t^2}\right] = v\left[\frac{3t - t^3}{1-t^2}\right] \qquad t = \tan\left(\frac{\alpha}{2}\right)$$

whereas $RHS = 2u \times \frac{2t}{1-t^2} = \frac{4ut}{1-t^2}$

So, as $LHS = RHS$, we obtain: $v(3t - t^3) = 4ut$

OR $v(3 - t^2) = 4u \Leftrightarrow u = \frac{v}{4}(3 - t^2)$

$t^2 > 0$ so $u < \frac{3v}{4}$