FUTHER APPLICATIONS OF SERIES

Annuities

An **annuity** (from the same Latin word that gives us 'per annum' and 'annual') is a series of payments made at equal intervals of time: payments are traditionally once per year (hence 'annuity'), but may also be half-yearly, quarterly or at more frequent intervals.

In this course, an annuity is defined as a form of investment in which periodical equal contributions are made to or taken from the investment, with interest compounding at the conclusion of each period.

For example, a person may deposit the same amount of money each time period (e.g. year) into a superannuation fund or an investment account, with the interest compounding at the end of the time period. At some point in the future, the funds may be taken out of the account as a lump sum or as a regular payment or pension.

Example 31

Ian deposits $50 000 in an account which will earn interest at a rate of 3% p.a., paid monthly. Ian wishes to withdraw $200 per month.

(a) Set up a recurrence relation that models this account and gives the balance after n payments.

(b) What is the balance after two withdrawals?

(c) Explain what will happen to the balance if the interest was 5% per annum from the start.

(d) What monthly payment would keep the balance at $50 000? Calculate the answer for rates of 5% and 3%.

Solution

(a) Amount deposited $= A_0 = \$50\,000$.

Interest rate per month $= r = \frac{3}{1200} = 0.0025$ so $R = 1 + r = 1.0025$

Time period $= n$ months so that A_1 is the balance at the end of the first month after interest has been added and the first payment made.

Monthly payment: $M = \$200$

After 1 month: $A_1 = 50\,000 \times 1.0025 - 200$

This can be written as: $A_1 = A_0 \times R - M$

$= A_0 \times 1.0025 - 200$

After 2 months: $A_2 = A_1 \times 1.0025 - 200$

After n months (and n payments) this becomes: $A_n = A_{n-1} \times 1.0025 - 200$

(b) $A_1 = 50\,000 \times 1.0025 - 200$

$= 49\,925$

$A_2 = 49\,925 \times 1.0025 - 200$

$= 49\,849.81$

After two withdrawals, the balance remaining is $49 849.81.

If the payment made each month is greater than the interest earned, then the value of A_n decreases as n increases and eventually $A_n \to 0$, with the last payment most likely being less than $200.

(c) $r = 5\%$ p.a., $r = \frac{5}{1200}$ p.m. $= \frac{1}{240}$ p.m.

$R = 1 + \frac{1}{240} = \frac{241}{240}$

$A_1 = 50\,000 \times \frac{241}{240} - 200$

$= 50\,008.33$

After the first payment, the balance remaining is larger than the initial deposit, so the amount in the account will continue to grow. At an interest rate of 5% p.a., the account earns more interest than the amount withdrawn each month.

(d) $\$50\,000 \times \frac{5}{100} \times \frac{1}{12} \approx \$208.33$

A monthly payment of $208.33 will leave $50 000 in the account if the annual interest rate remained at 5%.

$\$50\,000 \times \frac{3}{100} \times \frac{1}{12} = \$125$

If the interest rate is only 3%, then a monthly payment of $125 will leave $50 000 in the account. This means that for the account balance to remain unchanged, the payment taken each time period must be the same as the interest earned during that time period.

Annuities are usually designed to reduce in value over time. It is possible to calculate the payments to be made on a regular basis to meet individual needs. Of course, if the interest rate falls then the payments will cease earlier and if the interest rate rises then the payments will last longer.

Example 32

Sam lends \$5000 on the condition that she is repaid the money, plus interest, in 10 equal quarterly instalments. The first amount is to be repaid 3 months from the loan date. If this loan earns interest at a rate of 4% per quarter, what is the amount of each instalment?

Solution

$P = 5000$, $r = 0.04$, $R = 1.04$, repayment amount (instalment) is Q per quarter

Amount owing after first repayment: $= 5000 \times 1.04 - Q$

Amount owing after second repayment: $= (5000 \times 1.04 - Q) \times 1.04 - Q$

$= 5000 \times 1.04^2 - Q(1 + 1.04)$

Amount owing after third repayment: $= (5000 \times 1.04^2 - Q(1 + 1.04)) \times 1.04 - Q$

$= 5000 \times 1.04^3 - Q(1 + 1.04 + 1.04^2)$

Amount owing after tenth repayment: $= 5000 \times 1.04^{10} - Q(1 + 1.04 + 1.04^2 + \ldots + 1.04^9)$

Now $1 + 1.04 + 1.04^2 + \ldots + 1.04^9$ is a geometric series with $a = 1$, $r = 1.04$, $n = 10$.

After the tenth repayment, the amount owing is zero, so $A_{10} = 0$.

$\therefore$ Amount owing after tenth repayment: $5000 \times 1.04^{10} - \dfrac{Q\left(1.04^{10} - 1\right)}{1.04 - 1} = 0$

Hence: $\dfrac{Q\left(1.04^{10} - 1\right)}{1.04 - 1} = 5000 \times 1.04^{10}$

$$Q = \frac{5000 \times 1.04^{10} \times 0.04}{1.04^{10} - 1} = 616.45$$

Thus Sam receives \$616.45 per quarter.

This result can be generalised into a formula if you let $P = 5000$, $R = 1.04$. with n repayments:

Amount owing after n repayments $= P \times R^n - \dfrac{Q\left(R^n - 1\right)}{R - 1} = 0$

So repayments are: $Q = \dfrac{PR^n(R - 1)}{R^n - 1}$

Although this formula could have been used from the start, it is important to understand how it is obtained.

Example 33

When Peggy started work she began paying \$120 at the beginning of each month into a superannuation fund. These contributions are compounded monthly at an interest rate of 6% p.a. Peggy intends to retire after having worked for 40 years.

(a) Let $\$P$ be the final value of Peggy's superannuation when she retires after 40 years (480 months). Show that $\$P = \$240\,174$ to the next dollar.

(b) After working for 20 years, Peggy decides that she needs to have \$750 000 in her fund before she can retire. At this stage the fund has only \$55 722. Peggy decides to increase the amount that she pays into the fund for the next 20 years to $\$M$ at the beginning of each month. The contributions will continue to attract the same interest rate of 6% p.a. compounded monthly. At the end of n months after starting the new contributions, the amount in the fund is $\$A_n$.

(i) Show that $A_2 = 55\,722 \times 1.005^2 + M(1.005 + 1.005^2)$

(ii) Find the value of M so that Peggy will have \$750 000 in her fund after the remaining 20 years (240 months).

Solution

(a) Monthly investment = \$120, $r = 0.005\%$ p.m., $n = 480$

Value of last investment at end of 40th year:	$= 120 \times 1.005$	(1 month's interest)
Value of second-last investment at end of 40th year:	$= 120 \times 1.005^2$	(2 months' interest)
Value of first investment at end of 40th year:	$= 120 \times 1.005^{480}$	(480 months' interest)

Sum of the investments $= 120 \times 1.005 + 1200 \times 1.005^2 + 120 \times 1.005^3 + \ldots + 120 \times 1.005^{480}$

This is a finite geometric series with first term 120×1.005, common ratio 1.005, $n = 480$.

$$P = S_{480} = \frac{a\left(R^{480} - 1\right)}{R - 1}$$

$$= \frac{120 \times 1.005\left(1.05^{480} - 1\right)}{1.005 - 1} = \frac{120 \times 1.005\left(1.005^{480} - 1\right)}{0.005} = 240173.78$$

$\$P = \$240\,174$ (to the next dollar)

(b) (i) Monthly investment $= \$M$, $r = 0.005\%$ p.m., $n = 2$, initial balance = \$55 722

Value of balance at end of 2 months:	$= 55\,722 \times 1.005^2$
Value of last investment at end of 1 month:	$= M \times 1.005$
Value of second-last investment at end of 2 months:	$= M \times 1.005^2$

$\therefore A_2 = 55722 \times 1.005^2 + M(1.005 + 1.005^2)$

(ii) $A_{240} = 55\,722 \times 1.005^{240} + M(1.005 + 1.005^2 + 1.005^3 + \ldots + 1.005^{240})$

$$A_{240} = 55\,722 \times 1.005^{240} + \frac{M \times 1.005\left(1.005^{240} - 1\right)}{1.005 - 1}$$

We require $A_{240} = 750\,000$:

$$750000 = 55722 \times 1.005^{240} + \frac{M \times 1.005\left(1.005^{240} - 1\right)}{0.005}$$

$$750000 - 55722 \times 1.005^{240} = \frac{M \times 1.005\left(1.005^{240} - 1\right)}{0.005}$$

$$M = \frac{0.005\left(750000 - 55722 \times 1.005^{240}\right)}{1.005\left(1.005^{240} - 1\right)} = 1217.93$$

The monthly payment would need to be \$1218 per month (to the next dollar).

Example 34

An investment of \$20 000 earns interest at a rate of 2.7% p.a. compounded annually.

(a) What is the future value of this investment after 8 years?

(b) Use a recurrence relation to set up a spreadsheet to model this investment over 10 years.

Solution

(a) $P = 20\,000$, $r = 2.7\% = 0.027$. $R = 1.027$

$$A_8 = PR^8$$
$$= 20\,000 \times 1.027^8$$
$$= 24\,751.05$$

The future value of this investment is \$24 751.05.

(b) Set up the recurrence relation: $A_n = A_{n-1} \times 1.027$

	A	B	C	D	E
1	*P* =	20000	*R* =	1.027	*A* = *PR*^*n*
2	Amount 1 =	\$20,540.00			
3	Amount 2 =	\$21,094.58			
4	Amount 3 =	\$21,664.13			
5	Amount 4 =	\$22,249.07			
6	Amount 5 =	\$22,849.79			
7	Amount 6 =	\$23,466.73			
8	Amount 7 =	\$24,100.34			
9	Amount 8 =	\$24,751.05			
10	Amount 9 =	\$25,419.32			
11	Amount 10 =	\$26,105.65			

Calculating this with a spreadsheet shows that $A_8 = \$24\,751.05$ (which matches the figure calculated in **(a)**) and that $A_{10} = \$26\,105.65$.

Verifying the table of future value of an annuity

Annuity

An annuity is a compound interest investment from which equal payments are made or received on a regular basis (at equal periods of time) for a fixed period of time.

The payment is usually made at the end of the time period so that no interest is received until the end of the second time period.

Future value, FVA

The future value of an investment or annuity is the total value of the investment at the end of the term of investment, including all contributions and the interest earned.

At the start of the chapter, the formula used to obtain the FVA was given as $FVA = a\left\{\frac{(1+r)^n - 1}{r}\right\}$, where

FVA is the Future value of an annuity

a is the contribution per period paid at the end of the period

r is the interest rate per compounding period, as a decimal

n is the number of periods.

The formula was used to obtain the values in the table.

Consider an annuity in which 6 annual payments each of \$500 are made into an account where the compound interest rate is 4% per annum. The payments are made at the end of each year.

This information is shown in the following table.

Year	Balance at beginning of year (\$)	Interest on balance (\$)	Annual payment (\$)	Total at end of year (\$)
1	0.00	0.00	500	500.00
2	500.00	20.00	500	1020.00
3	1020.00	40.80	500	1560.80
4	1560.80	62.43	500	2123.23
5	2123.23	84.93	500	2708.16
6	2708.16	108.33	500	3316.49

$$
\begin{aligned}
FVA &= 500 + 500 \times 1.04 + 500 \times 1.04^2 + 500 \times 1.04^3 + 500 \times 1.04^4 + 500 \times 1.04^5 \\
&= 500(1 + 1.04 + 1.04^2 + 1.04^3 + 1.04^4 + 1.04^5) \\
&= \frac{500\left(1.04^6 - 1\right)}{1.04 - 1}, \text{ by summing a geometric series with 6 terms} \\
&= \$3316.49
\end{aligned}
$$

If there had been n annual payments of \$$a$ made at the end of each year with a compound interest rate of r% per year, then if you would have

$$
\begin{aligned}
FVA &= a(1 + (1 + r) + (1 + r)^2 + \ldots + (1 + r)^{n-1}) \\
&= a\left\{\frac{(1+r)^n - 1}{r}\right\}
\end{aligned}
$$

This verifies the formula that was used to obtain the FVA table used at the beginning of the chapter.