INTEGRALS OF THE TYPE f'(x) (f(x))n

1 Find: **(a)** $\int \sin x \cos^2 x\, dx$ **(b)** $\int \tan x \sec^2 x\, dx$ **(c)** $\int \sin x \cos^3 x\, dx$

a) $u = \cos x$ so $\frac{du}{dx} = -\sin x$ so $du = -\sin x\, dx$

$\int \sin x \cos^2 x\, dx = \int \cos^2 x \times \sin x\, dx = -\int \cos^2 x \times (-\sin x\, dx) = -\int u^2 du$

$\text{———} = -\int u^2\, du = -\frac{u^3}{3} + C = -\frac{\cos^3 x}{3} + C$

b) $u = \tan x$ $\frac{du}{dx} = \sec^2 x$ so $du = \sec^2 x\, dx$

$\int \tan x \sec^2 x\, dx = \int u \times du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$

c) $u = \cos x$ $\frac{du}{dx} = -\sin x$ so $du = -\sin x\, dx$

$\int \sin x \cos^3 x\, dx = \int \cos^3 x \times \sin x\, dx = -\int \cos^3 x \times (-\sin x\, dx)$

$\text{———} = -\int u^3 \times du = -\frac{u^4}{4} + C$

$\text{———} = -\frac{\cos^4 x}{4} + C$

INTEGRALS OF THE TYPE f'(x) (f(x))n

1 Find: **(d)** $\int \cos x \sin^4 x\, dx$ **(e)** $\int (1+\cos 2x)\sin x\, dx$ **(f)** $\int \sin x \cos x\, dx$

d) $u = \sin x \qquad \frac{du}{dx} = \cos x \qquad$ so $du = \cos x\, dx$

$$\int \cos x \sin^4 x\, dx = \int \sin^4 x \times \cos x\, dx = \int u^4 \times du$$

$$= \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C$$

e) $\int (1+\cos 2x)\sin x\, dx = \int 2\cos^2 x \times \sin x\, dx = 2\int \cos^2 x \sin x\, dx$

as $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

$u = \cos x \qquad \frac{du}{dx} = -\sin x \qquad$ so $du = -\sin x\, dx$

$$2\int \cos^2 x \sin x\, dx = -2\int \cos^2 x\,(-\sin x\, dx) = -2\int u^2\, du$$

$$= -2\,\frac{u^3}{3} + C = -\frac{2}{3}\cos^3 x + C$$

f) $\int \sin x \cos x\, dx = \int \frac{\sin 2x}{2}\, dx = \frac{1}{2}\int \sin 2x\, dx$

$$= \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + C$$

$$= -\frac{1}{4}\cos 2x + C$$

OR $u = \sin x \qquad \frac{du}{dx} = \cos x \qquad du = \cos x\, dx$

$$\int \sin x \cos x\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$$

both are equal as $\cos 2x = 1 - 2\sin^2 x$

INTEGRALS OF THE TYPE f'(x) (f(x))n

2 Evaluate: **(a)** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \cos^3 x \, dx$ **(c)** $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$

a) $u = \cos x$ $\quad \frac{du}{dx} = -\sin x$ $\quad$ so $du = -\sin x \, dx$

when $x = \pi/4$ $\quad u = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

when $x = \pi/2$ $\quad u = \cos\frac{\pi}{2} = 0$

So $\int_{\pi/4}^{\pi/2} \sin x \cos^3 x \, dx = -\int_{\pi/4}^{\pi/2} \cos^3 x \, (-\sin x \, dx) = -\int_{\sqrt{2}/2}^{0} u^3 \, du = \int_0^{\sqrt{2}/2} u^3 \, du$

$$= \left[\frac{u^4}{4}\right]_0^{\sqrt{2}/2} = \frac{1}{4} \times \left(\frac{\sqrt{2}}{2}\right)^4 = \frac{1}{4} \times \frac{4}{16} = \frac{1}{16}$$

Alternatively: you find the primitive first, and then find the integral.

b) $u = \tan x$ $\quad \frac{du}{dx} = \sec^2 x$ $\quad$ so $du = \sec^2 x \, dx$

$$\int \tan x \sec^2 x \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$$

So $\int_0^{\pi/4} \tan x \sec^2 x \, dx = \left[\frac{\tan^2 x}{2}\right]_0^{\pi/4} = \frac{1}{2}\left[\tan^2\frac{\pi}{4} - \tan^2 0\right]$

$$= \frac{1}{2} \times 1 = \frac{1}{2}$$

INTEGRALS OF THE TYPE f'(x) (f(x))n

2 Evaluate: (e) $\int_0^{\pi} 2\sin\theta\cos^2\theta\, d\theta$ (f) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\left(x-\frac{\pi}{4}\right)dx$

e) $u = \cos\theta$ $\quad \frac{du}{d\theta} = -\sin\theta$ $\quad du = -\sin\theta\, d\theta$

$$\int 2\sin\theta\cos^2\theta\, d\theta = -2\int \cos^2\theta \times (-\sin\theta)\, d\theta = -2\int u^2\, du = -2\frac{u^3}{3} + C$$

$$= -\frac{2}{3}\cos^3\theta + C \text{, so}$$

$$\int_0^{\pi} 2\sin\theta\cos^2\theta\, d\theta = \left[-\frac{2}{3}\cos^3\theta\right]_0^{\pi} = \left[\frac{2}{3}\cos^3\theta\right]_{\pi}^{0} = \frac{2}{3}\left[\cos^3\theta\right]_{\pi}^{0}$$

$$= \frac{2}{3}\left[\cos^3 0 - \cos^3\pi\right] = \frac{2}{3}\left[1 - (-1)^3\right] = \frac{2}{3}\times 2 = \frac{4}{3}$$

f)

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \text{ so } \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int_{-\pi/2}^{\pi/2} \cos^2\left(x-\frac{\pi}{4}\right)dx = \int_{-\pi/2}^{\pi/2} \frac{1+\cos(2x-\pi/2)}{2}\, dx$$

$$= \frac{1}{2}\left[\int_{-\pi/2}^{\pi/2} dx + \int_{-\pi/2}^{\pi/2} \cos\left(2x-\frac{\pi}{2}\right)dx\right]$$

But $\cos\left(\alpha - \frac{\pi}{2}\right) = \sin\alpha$ so

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) + \int_{-\pi/2}^{\pi/2} \sin 2x\, dx\right]$$

$$= \frac{1}{2}\left[\pi + \left[-\frac{\cos 2x}{2}\right]_{-\pi/2}^{\pi/2}\right] = \frac{\pi}{2} + \frac{1}{4}\left[\cos 2x\right]_{\pi/2}^{-\pi/2}$$

$$= \frac{\pi}{2} + \frac{1}{4}\left[\cos(-\pi) - \cos\pi\right] = \frac{\pi}{2}$$