INTEGRALS INVOLVING LOGARITHMIC FUNCTIONS

You have studied the derivative of the logarithmic function in the Mathematics Advanced course (see *New Senior Mathematics Advanced for Years 11 & 12*, Chapter 13). For example:

$\frac{d}{dx}\log_e(x^3+1) = \frac{3x^2}{x^3+1}$ $\qquad$ $\frac{d}{dx}\log_e(x^2+2x+1) = \frac{2x+2}{x^2+2x+1}$ $\qquad$ $\frac{d}{dx}\log_e(\sin x) = \frac{\cos x}{\sin x}$, $\sin x > 0$

These results are all of the form $\frac{d}{dx}\log_e(f(x)) = \frac{f'(x)}{f(x)}$.

Consider $f(x) = \log_e[h(x)]$. Let $u = h(x)$ so $\frac{du}{dx} = h'(x)$:

$$\begin{aligned} f'(x) &= \frac{1}{h(x)} \times \frac{du}{dx} \\ &= \frac{h'(x)}{h(x)} \end{aligned}$$

Hence it follows that:

$$\int \frac{dx}{x} = \log_e x + C, x > 0$$

$$\int \frac{h'(x)}{h(x)} dx = \log_e[h(x)] + C, h(x) > 0$$

Important consideration:

The function where $f(x) = \log_e(-x)$ is a logarithmic function defined for $x < 0$.

Let $u = -x$ so that $y = \log_e u$: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ now gives $\frac{dy}{dx} = \frac{1}{u} \times (-1) = \frac{-1}{u} = \frac{-1}{-x} = \frac{1}{x}$

This is the same result as previously obtained for the function defined for $x > 0$.

This means that: $$\int \frac{dx}{x} = \begin{cases} \log_e x + C, & x > 0 \\ \log_e(-x) + C, & x < 0 \end{cases}$$

i.e. $$\int \frac{dx}{x} = \log_e|x| + C$$

Example 11

Evaluate: **(a)** $\int_1^2 \frac{x}{1+x^2} dx$ $\qquad$ **(b)** $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{1-\cos x} dx$ $\qquad$ **(c)** $\int_0^1 \frac{e^x}{1+e^x} dx$

Solution

(a) $$\begin{aligned} \int_1^2 \frac{x}{1+x^2} dx &= \frac{1}{2}\int_1^2 \frac{2x}{1+x^2} dx \\ &= \frac{1}{2}\left[\log_e(1+x^2)\right]_1^2 \\ &= \frac{1}{2}(\log_e 5 - \log_e 2) \\ &= \frac{1}{2}\log_e 2.5 \end{aligned}$$

(b) $$\begin{aligned} \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{1-\cos x} dx &= \left[\log_e(1-\cos x)\right]_{\frac{\pi}{2}}^{\pi} \\ &= \log_e 2 - \log_e 1 \\ &= \log_e 2 \end{aligned}$$

(c) $$\begin{aligned} \int_0^1 \frac{e^x}{1+e^x} dx &= \left[\log_e(1+e^x)\right]_0^1 \\ &= \log_e(1+e) - \log_e 2 \\ &= \log_e \frac{1+e}{2} \end{aligned}$$

Example 10

Write a primitive (antiderivative) of each function, stating any restrictions that must be placed on x.

(a) $\frac{4}{2x-3}$ **(b)** $\frac{6x^2+3}{2x^3+3x-5}$ **(c)** $\frac{\sin x}{1+\cos x}$ **(d)** $\frac{x}{x-1}$ **(e)** $\frac{x^2}{x-1}$

Solution

(a) $\int \frac{4}{2x-3}dx = 2\int \frac{2}{2x-3}dx = 2\int \frac{h'(x)}{h(x)}dx$ where $h(x) = 2x-3$

$= 2\log_e(2x-3) + C, x > 1.5$

or $2\log_e|2x-3| + C$ without the restriction on x.

(b) $\int \frac{6x^2+3}{2x^3+3x-5}dx = \int \frac{h'(x)}{h(x)}dx$ where $h(x) = 2x^3+3x-5$

$= 2\log_e(2x^3+3x-5) + C, x > 1$

It is not obvious that $2x^3+3x-5 > 0$ when $x > 1$, so it is better to write this answer as:

$= 2\log_e|2x^3+3x-5| + C$

(c) $\int \frac{\sin x}{1+\cos x}dx = -\int \frac{-\sin x}{1+\cos x}dx = -\int \frac{h'(x)}{h(x)}dx$ where $h(x) = 1+\cos x$

$= -\log_e(1+\cos x) + C, \cos x \neq -1$

or $= -\log_e|1+\cos x| + C$

(d) The power of the numerator equals the power of the denominator, so division gives:

$\frac{x}{x-1} = \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1}$

$\therefore \int \frac{x}{x-1}dx = \int\left(1+\frac{1}{x-1}\right)dx = x + \ln|x-1| + C$

(e) The power of the numerator is greater than the power of the denominator, so division gives:

$\frac{x^2}{x-1} = \frac{(x^2-x)+(x-1)+1}{x-1} = x+1+\frac{1}{x-1}$ Long division requires $x-1\overline{)x^2+0x+0}$.

$\therefore \int \frac{x^2}{x-1}dx = \int\left(x+1+\frac{1}{x-1}\right)dx = \frac{x^2}{2} + x + \log_e(x-1) + C, x > 1$

or $\frac{x^2}{2} + x + \log_e|x-1| + C$