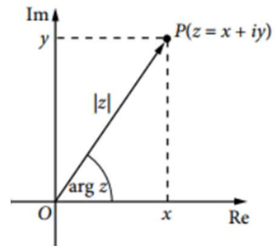
GEOMETRIC REPRESENTATION OF A COMPLEX NUMBER AS A VECTOR

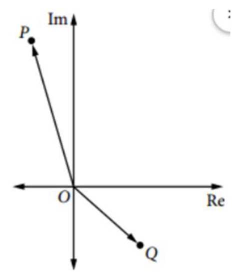
A vector is a mathematical object that has both magnitude and direction.

Complex numbers can be represented as vectors for which the modulus gives the magnitude and the argument gives the direction.

On an Argand diagram, let point P represent the complex number $z = x + i\,y$ (see diagram at right). This z is also represented by the vector $\overrightarrow{OP}$, where $\overrightarrow{OP}$ has length $|z|$ and direction given by a rotation from the positive direction of the real axis by $arg(z)$



In the diagram to the right, point P represents the complex number $-1 + 3i$ and point Q represents the complex number $2\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$.
Here the vector $\overrightarrow{OP}$ represents $-1 + 3i$ and the vector $\overrightarrow{OQ}$ represents $2\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$.



The vector to represent a complex number does not have to 'start' at the origin O.

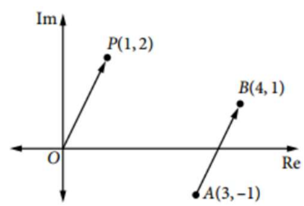
For example, the number $1 + 2i$ could be represented by any vector with length $\sqrt{5}$ that is inclined at an angle $\tan^{-1} 2$.

Both $\overrightarrow{OP}$ and $\overrightarrow{AB}$ represent $1 + 2i$

Point P represents $1 + 2i$, but point B does not.

$\overrightarrow{OP}$ is called the **position vector.**

$\overrightarrow{AB}$ is called a **free vector.**



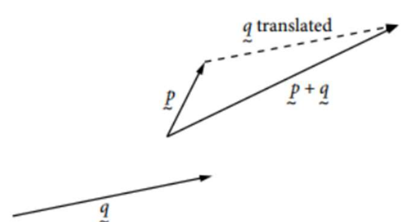
Note: $\overrightarrow{BA}$ does not represent $1 + 2i$. In fact, $\overrightarrow{BA}$ represent $-1 - 2i$

Operations on complex numbers represented as vectors

Addition (to form the vector sum $\underset{\sim}{p} + \underset{\sim}{q}$):

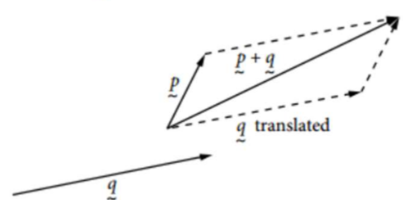
Method 1 (complete the triangle)

1. Translate $\underset{\sim}{q}$ so that its tail is located at the head of $\underset{\sim}{p}$.
2. The vector for the sum $\underset{\sim}{p} + \underset{\sim}{q}$ now goes from the tail of $\underset{\sim}{p}$ to the head of the translated $\underset{\sim}{q}$.



Method 2 (construct the diagonal of the parallelogram)

1. Translate $\underset{\sim}{q}$ so that its tail is located at the tail of $\underset{\sim}{p}$.
2. Locate the fourth vertex of the parallelogram of vectors.
3. The vector for the sum $\underset{\sim}{p} + \underset{\sim}{q}$ now goes from the common tail of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ translated to the fourth vertex of the parallelogram.



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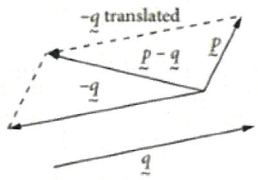
Subtraction (to form the vector difference $\underset{\sim}{p}-\underset{\sim}{q}$):

Method 1 (add the opposite vector)

1 Form the vector $-\underset{\sim}{q}$.

2 Form the vector sum $\underset{\sim}{p}+(-\underset{\sim}{q})$.

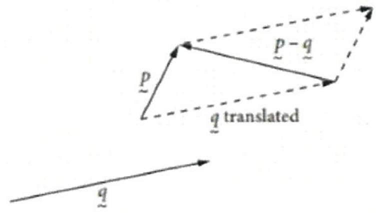


Method 2 (construct the other diagonal of the parallelogram)

1 Translate $\underset{\sim}{q}$ so that its tail is located at the tail of $\underset{\sim}{p}$.

2 Locate the fourth vertex of the parallelogram of vectors.

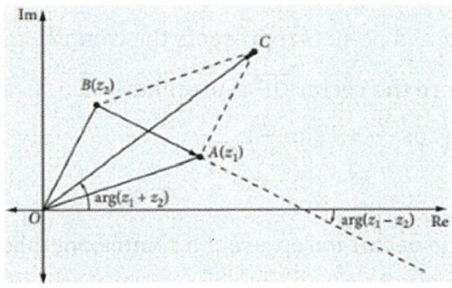
3 The vector for the sum $\underset{\sim}{p}-\underset{\sim}{q}$ is the diagonal of the parallelogram that goes *from* the head of $\underset{\sim}{q}$ *to* the head of $\underset{\sim}{p}$.



Let A represents z_1 and B represents z_2.

When the parallelogram $OACB$ is formed:

- vector $\overrightarrow{OC}$ represents $z_1 + z_2$
- vector $\overrightarrow{AB}$ represents $z_2 - z_1$ and vector $\overrightarrow{BA}$ represents $z_1 - z_2$
- $|z_1 + z_2|$ and $|z_1 - z_2|$ are the lengths of diagonals of the parallelogram
- $arg(z_1 + z_2)$ and $arg(z_1 - z_2)$ are the angles at which the diagonals are inclined to the positive direction of the real axis.



Example 27

On an Argand diagram, let A represent $z_1 = 3 + 2i$ and B represent $z_2 = -2 + 4i$. Show the following:

(a) a point C that represents $z_1 + z_2$

(b) a vector that represents $z_1 + z_2$

(c) a vector that represents $z_1 - z_2$

(d) a point D that represents $z_1 - z_2$

(e) a point E that represents $\frac{1}{2}(z_1 + z_2)$.

(f) What complex number is represented by the vector $\overrightarrow{EB}$?

(g) What geometrical relationships are there between the intervals EB and DO?

Solution

(a) Locate C by completing the parallelogram.

(b) $\overrightarrow{OC}$

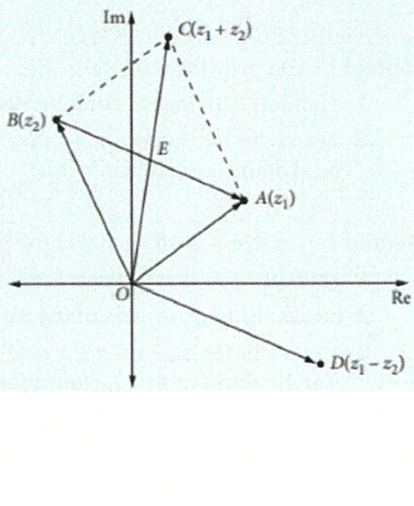
(c) $\overrightarrow{BA}$

(d) D is at the head of $\overrightarrow{OD}$, which is equal to $\overrightarrow{BA}$.

(e) E is the midpoint of the diagonal OC.

(f) $z_2 - \frac{1}{2}(z_1 + z_2) = \frac{1}{2}(z_2 - z_1)$

(g) $EB = \frac{1}{2}DO$, $EB \parallel DO$



GEOMETRIC REPRESENTATION OF A COMPLEX NUMBER AS A VECTOR

Multiplication by i

When working with vector representations of complex numbers, multiplication by i rotates the vector anticlockwise by $\frac{\pi}{2}$

Multiplication by ki (where k is real)

To multiply by ki, rotate the vector anticlockwise by $\frac{\pi}{2}$ and scale by a factor (dilation) of k

Multiplication of a complex number z by another complex number $r(\cos\theta + i\sin\theta) = re^{i\theta}$

To multiply z by $re^{i\theta}$, rotate the vector that represents z through an angle θ and scale by a factor of r

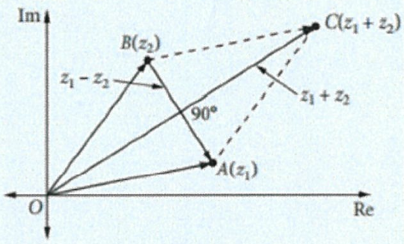
Example 28

If k is real and $\frac{z_1 - z_2}{z_1 + z_2} = ki$, show that $|z_1| = |z_2|$.

Solution

Show the problem on an Argand diagram. Let A represent z_1 and B represent z_2. Complete the parallelogram $OACB$ such that C represents $z_1 + z_2$.

As a result of this, the vectors along the diagonals of the parallelogram represent $z_1 + z_2$ (vector $\overrightarrow{OC}$) and $z_1 - z_2$ (vector $\overrightarrow{BA}$).



As $\frac{z_1 - z_2}{z_1 + z_2} = ki$, thus $\arg\left(\frac{z_1 - z_2}{z_1 + z_2}\right) = \arg(ki)$

$$\arg(z_1 - z_2) - \arg(z_1 + z_2) = \pm\frac{\pi}{2}$$

i.e. the diagonals meet at right angles.

Hence $OACB$ is a rhombus (parallelogram with perpendicular diagonals).

$\therefore |z_1| = |z_2|$ (sides of a rhombus are equal)

<u>Hint</u>: whenever you see $z_1 + z_2$ and $z_1 - z_2$ in a problem, the problem can most likely be solved using the geometrical properties of parallelograms.

The triangle inequalities

In a triangle OAC, $OC \leq OA + AC$, with $OC = OA + AC$ when the points O, A and C are colinear.

$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$ with $|z_1 + z_2| = |z_1| + |z_2|$ when $z_2 = k\,z_1$ (where k is real).

In triangle OAB: $OA \leq OB + BA$, with equality when the points O, A and B are colinear.

$\therefore |z_1| \leq |z_2| + |z_1 - z_2|$

$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$ with equality when $z_2 = k\,z_1$ (where k is real).