OTHER USEFUL TECHNIQUES

Properties of the definite integral

1 $\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b \big(f(x) \pm g(x)\big)\,dx$

2 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

3 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4 $\int_a^b f(x)\,dx = \int_a^b f(u)\,du$ (i.e. the value of the integral is independent of the variable of integration)

5 If $f(-x) = f(x)$ (i.e. if $f(x)$ is an **even** function), then: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

6 If $f(-x) = -f(x)$ (i.e. if $f(x)$ is an **odd** function), then: $\int_{-a}^{a} f(x)\,dx = 0$.

7 $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$ represents the area of a semicircle of radius a, so it can be quickly evaluated using $A = \pi r^2$

Combinations of even and odd functions

- The product of two **odd** functions or two **even** functions is an **even** function.
- The product of an **odd** function and an **even** function is an **odd** function.

Example 32

Without evaluating the definite integral, find a simpler expression for each definite integral.

(a) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x\,dx$ **(b)** $\int_{-1}^{1} x \sin^{-1} x\,dx$

Solution

(a) $f(x) = x^3$ is an odd function.
$g(x) = \cos x$ is an even function.
Hence $f(x) \times g(x) = [\text{odd}] \times [\text{even}] = [\text{odd}]$

$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x\,dx = 0$

(b) $f(x) = x$ is an odd function.
$g(x) = \sin^{-1} x$ is an odd function.
Hence $f(x) \times g(x) = [\text{odd}] \times [\text{odd}] = [\text{even}]$

$\therefore \int_{-1}^{1} x \sin^{-1} x\,dx = 2\int_0^1 x \sin^{-1} x\,dx$

Example 34

Evaluate: $\int_{-\pi}^{\pi} \sin^5 x \cos^8 x\,dx$

Solution

This looks formidable, but the properties of odd and even functions make it quite simple to evaluate.

$f(x) = \sin^5 x$ is an odd function, $g(x) = \cos^8 x$ is an even function:
$f(x) \times g(x) = [\text{odd}] \times [\text{even}] = [\text{odd}]$

Hence: $\int_{-\pi}^{\pi} \sin^5 x \cos^8 x\,dx = 0$

Example 33

Use the properties of odd and even functions to evaluate: $\int_{-1}^{1}(1+x^3)^3\,dx$

Solution

$f(x)=(1+x^3)^3 \qquad f(-x)=(1-x^3)^3 \qquad \therefore f$ is neither odd nor even.

Expand: $(1+x^3)^3 = 1+3x^3+3x^6+x^9$

$$\begin{aligned}
\therefore \int_{-1}^{1}(1+x^3)^3\,dx &= \int_{-1}^{1}(1+3x^3+3x^6+x^9)\,dx \\
&= \int_{-1}^{1}dx + 3\int_{-1}^{1}x^3\,dx + 3\int_{-1}^{1}x^6\,dx + \int_{-1}^{1}x^9\,dx \\
&= [\text{even}] + [\text{odd}] + [\text{even}] + [\text{odd}] \\
&= 2\int_{0}^{1}dx + 0 + 2\times 3\int_{0}^{1}x^6\,dx + 0 \\
&= 2\int_{0}^{1}(1+3x^6)\,dx \\
&= 2\left[x+\frac{3x^7}{7}\right]_0^1 = 2\left(1+\frac{3}{7}\right) = \frac{20}{7}
\end{aligned}$$

Example 35

(a) Show that: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(b) Hence evaluate: $\int_0^{\pi} x\sin^2 x\,dx$

Solution

(a) Method 1

If $\int f(x)\,dx = F(x)+C$ then $\int_0^a f(x)\,dx = [F(x)]_0^a = F(a)-F(0)$

and $\int_0^a f(a-x)\,dx = [-F(a-x)]_0^a = -F(0)+F(a) = F(a)-F(0) = \int_0^a f(x)\,dx$

Method 2

Let $y=a-x$ so that $dy=-dx$. Limits are $x=0$: $y=a$ $\qquad x=a$: $y=0$

$$\int_0^a f(a-x)\,dx = \int_a^0 f(y)(-dy) = \int_0^a f(y)\,dy = \int_0^a f(x)\,dx$$

(b) You can write: $\int_0^{\pi} x\sin^2 x\,dx = \int_0^{\pi}(\pi-x)\sin^2(\pi-x)\,dx$

$$\begin{aligned}
\text{Now} \quad \int_0^{\pi}(\pi-x)\sin^2(\pi-x)\,dx &= \int_0^{\pi}(\pi-x)\sin^2 x\,dx \\
&= \int_0^{\pi}\pi\sin^2 x\,dx - \int_0^{\pi}x\sin^2 x\,dx
\end{aligned}$$

$$\begin{aligned}
\text{Hence:} \quad \int_0^{\pi}x\sin^2 x\,dx &= \pi\int_0^{\pi}\sin^2 x\,dx - \int_0^{\pi}x\sin^2 x\,dx \\
2\int_0^{\pi}x\sin^2 x\,dx &= \frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)\,dx \\
\int_0^{\pi}x\sin^2 x\,dx &= \frac{\pi}{4}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi} \\
&= \frac{\pi}{4}(\pi-0) = \frac{\pi^2}{4}
\end{aligned}$$

Warning

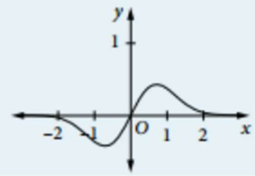
Be aware of the difference between *evaluating the integral* $\int_{-a}^{a} f(x)\,dx$ and *finding the area* under the curve $y=f(x)$ between the ordinates $x=-a$ and $x=a$.

Example 36

(a) Evaluate: $\int_{-2}^{2} xe^{-x^2}\,dx$

(b) Find the area bounded by the curve $y=xe^{-x^2}$, the x-axis and the ordinates $x=-2$ and $x=2$.

Solution



(a) xe^{-x^2} is an odd function, so: $\int_{-2}^{2} xe^{-x^2}\,dx = 0$

(b) Area $= \int_{-2}^{2} xe^{-x^2}\,dx = 2\int_{0}^{2} xe^{-x^2}\,dx = \left[-e^{-x^2}\right]_0^2 = 1-e^{-4}$ units^2

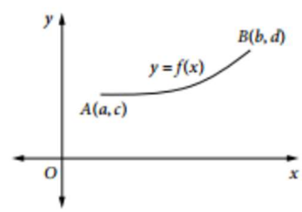
Formulae involving integration

The length of the arc of the curve $y=f(x)$ between $x=a$ and $x=b$ is given by:

$$\text{Arc length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_{u_1}^{u_2} \sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2}\,du$$



Here $u=u_1$ and $u=u_2$ are the parameters of A and B respectively.

Example 37

Calculate the arc length of the curve $y=\frac{x^4}{8}+\frac{1}{4x^2}$ from $x=1$ to $x=2$ using the formula:

$$\text{Arc length} = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Solution

$\frac{dy}{dx} = \frac{x^3}{2}-\frac{1}{2x^3}$ $\quad\therefore$ Arc length $= \int_1^2 \sqrt{1+\left(\frac{x^3}{2}-\frac{1}{2x^3}\right)^2}\,dx$

$$= \int_1^2 \sqrt{1+\frac{x^6}{4}-\frac{1}{2}+\frac{1}{4x^6}}\,dx$$

$$= \int_1^2 \sqrt{\frac{x^6}{4}+\frac{1}{2}+\frac{1}{4x^6}}\,dx$$

$$= \int_1^2 \sqrt{\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)^2}\,dx$$

$$= \int_1^2 \left(\frac{x^3}{2}+\frac{1}{2x^3}\right)dx = \left[\frac{x^4}{8}-\frac{1}{4x^2}\right]_1^2 = \frac{33}{16}$$