Quadratic equations with real coefficients for which the discriminant (i.e. $\Delta = b^2 - 4ac$) is negative

Example 20

(a) Solve $x^2 + x + 1 = 0$.

(b) Discuss the nature of the roots.

Solution

(a) Using the quadratic formula: $x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$.

Write the answer in the form $a + bi$: $x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

(b) The roots are $x = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $x = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$.

These complex numbers are of the form $a + bi$ and $a - bi$, that is they are conjugates.

<u>The complex roots of a quadratic equation with real coefficients occur as conjugate pairs.</u>

Proof: Consider the quadratic equation with real coefficients $ax^2 + bx + c = 0$

If the discriminant $\Delta = b^2 - 4ac$ is negative, then that means that $4ac - b^2$ is positive.

Therefore $\sqrt{\Delta}$ can be written as $\sqrt{\Delta} = \sqrt{-(4ac - b^2)} = \sqrt{4ac - b^2} \times \sqrt{-1} = \sqrt{4ac - b^2} \times i = k\,i$

The two solutions of the quadratic equation are therefore:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-b \pm k\,i}{2a} = \frac{-b}{2a} \pm i\,\frac{k}{2a}$$

So indeed the complex roots of a quadratic equation with real coefficients occur as conjugate pairs.

Polynomial equations

A polynomial $P(z)$ of degree n in one variable is an expression of the form $a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$.

In general, all polynomial equations $P(z) = 0$ of degree n have n solutions over the field of complex numbers. The proof of this statement is beyond the scope of this course.

If any of the coefficients $a_n, a_{n-1}, \ldots$ are complex numbers then the polynomial is a polynomial over the set of complex numbers. If all of the coefficients are real numbers, then the polynomial is also a polynomial over the set of real numbers.

- $z^2 - 5z + 1$ is a 2nd-degree polynomial over the set of real numbers, the set of complex numbers and the set of integers.
- $z^3 + iz^2 + (2 - 3i)z + 2$ is a 3rd-degree polynomial over the set of complex numbers, but not over the set of real numbers.

Remember that the set of complex numbers contains the sets of real numbers, rational numbers and integers, so that any polynomial over the reals, rationals or integers is also a polynomial over the complex numbers.

Note: Even if the coefficients are all real numbers, the solutions of the polynomial equation may involve complex numbers. For example, $z^2 - 4z + 5 = 0$ is a quadratic equation with integer coefficients; but the solutions for z are complex numbers (as $\Delta < 0$).

Fundamental theorem of algebra

Every polynomial equation with complex coefficients, $P(z) = 0$, of degree n (where n is a positive integer) has a root that is a complex number.

This important theorem was first proved convincingly by the German scientist Carl Friedrich Gauss (1777–1855), and then more completely and rigorously a few years later by the French mathematician Jean-Robert Argand. You can use it to show, with the aid of the factor theorem, that a polynomial of degree n is reducible to n linear factors and that a polynomial equation has no more than n roots:

- Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$.
- By the fundamental theorem of algebra, the equation $P(z) = 0$ has a root z_1 such that $P(z_1) = 0$.
- Hence, by the factor theorem, $(z - z_1)$ is a factor of $P(z)$.

Thus $P(z) = (z - z_1)Q_{n-1}(z) = 0$, where $Q_{n-1}(z)$ is a polynomial of degree $n - 1$.

- Applying the fundamental theorem of algebra again, the equation $Q_{n-1}(z) = 0$ similarly has a root z_2.

Thus $P(z) = (z - z_1)(z - z_2)Q_{n-2}(z) = 0$ where $Q_{n-2}(z)$ is a polynomial of degree $n - 2$.

By continuing this application of the fundamental theorem of algebra, after n applications you have:

$P(z) = a_n(z - z_1)(z - z_2)\ldots(z - z_n)$ where $a_n \neq 0$, as each factor is monic and the leading term of $P(z)$ is $a_n z^n$.

If z is any number different from $z_1, z_2, \ldots z_n$, then $P(z) \neq 0$. Thus $P(z)$ does not have any more than n zeros. (Note also that the complex numbers $z_1, z_2, \ldots z_n$ may not all be different from each other.)

Example 21

Reduce $z^4 + z^2 - 12$ to its linear factors over the complex numbers. Hence find the values of z for which $z^4 + z^2 - 12 = 0$.

Solution

$$
\begin{aligned}
z^4 + z^2 - 12 &= (z^2 - 3)(z^2 + 4) \\
&= (z - \sqrt{3})(z + \sqrt{3})(z - 2i)(z + 2i)
\end{aligned}
$$

Hence the roots of $z^4 + z^2 - 12 = 0$ are $z = \sqrt{3}, -\sqrt{3}, 2i, -2i$.

Notice that $z^4 + z^2 - 12$ is reduced to $(z^2 - 3)(z^2 + 4)$ over the rational numbers, and to $(z - \sqrt{3})(z + \sqrt{3})(z^2 + 4)$ over the real numbers.

Example 22

Reduce $2z^3 - 3z^2 + 8z + 5$ to its linear factors over the set of complex numbers. Hence find the values of z for which $2z^3 - 3z^2 + 8z + 5 = 0$.

Solution

$P(z) = 2z^3 - 3z^2 + 8z + 5$

If $(az - b)$ is a factor of $P(z)$, then you know from the factor theorem that $P\left(\frac{b}{a}\right) = 0$.

The coefficients of $P(z)$ are integers, so if there is to be a rational zero then b must be a factor of 5 and a must be a factor of 2. Thus the only possible values for $\frac{b}{a}$ are $\pm 1, \pm 5, \pm\frac{1}{2}, \pm\frac{5}{2}$.

Substitution of these values shows that the integer values do not work. However:

$$P\left(-\frac{1}{2}\right) = -\frac{1}{4} - \frac{3}{4} - 4 + 5 = 0$$

Hence $(2z + 1)$ is a factor of $P(z)$.

$$\begin{array}{r|l} & z^2 - 2z + 5 \\ \hline 2z + 1 & 2z^3 - 3z^2 + 8z + 5 \\ & \underline{2z^3 + z^2} \\ & -4z^2 + 8z \\ & \underline{-4z^2 - 2z} \\ & 10z + 5 \\ & \underline{10z + 5} \\ & 0 \end{array}$$

Thus $P(z) = (2z + 1)(z^2 - 2z + 5)$.

Completing the square:

$$\begin{aligned} P(z) &= (2z + 1)\left[(z^2 - 2z + 1) + 4\right] \\ &= (2z + 1)\left[(z - 1)^2 + 4\right] \\ &= (2z + 1)\left[(z - 1)^2 - 4i^2\right] \\ &= (2z + 1)[(z - 1 - 2i)(z - 1 + 2i)] \end{aligned}$$

Hence $2z^3 - 3z^2 + 8z + 5 = 0$ when $z = -\frac{1}{2}, 1 + 2i, 1 - 2i$.

Notice that $z = -\frac{1}{2}$ is a real root and that $z = 1 + 2i$, $z = 1 - 2i$ are a conjugate pair (i.e. their sum and product are real).

Only simple cases involving following factorisation techniques that were used before will be considered:

- quadratic trinomials
- sum and difference of two squares $z^2 \pm a^2$
- sum and difference of two cubes $z^3 \pm a^3$
- factor theorem (with at least one rational zero that can be found by trial and error)
- grouping