INDEFINITE INTEGRALS AND SUBSTITUTION

Some integrals can only be solved using particular substitutions for the variables. In this Mathematics Extension 1 course, any substitutions needed to find an integral are given.

Integration using a substitution can be considered as the converse of the method of differentiating a composite function—it's like using the chain rule backwards.

The aim of a substitution is to transform an integral into one that involves a standard result, e.g. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$. Variable substitution works as follows:

$$\text{Let} \quad y = \int f(u)du \quad \text{where } u = g(x)$$

$$\therefore \quad \frac{dy}{du} = f(u)$$

$$\text{But} \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= f(u) \times \frac{du}{dx}$$

$$\therefore \quad y = \int f(u) \times \frac{du}{dx} dx$$

$$\int f(u) \times \frac{du}{dx} dx = \int f(u)du$$

This 'backwards' form of the chain rule is convenient when the substitution of $u = g(x)$ allows a function to be expressed as the product of $\frac{du}{dx}$ and a function of u. For example:

- If $f(x) = 2x^2(x^3 - 1)^4$ then you can substitute $u = x^3 - 1$. As $\frac{du}{dx} = 3x^2$, you can write $2x^2$ as $\frac{2}{3} \times 3x^2$, so that $f(x)$ is written: $f(x) = \frac{2}{3}(x^3 - 1)^4 \times (3x^2)$
  $$= \frac{2}{3}u^4 \frac{du}{dx} \quad \text{where } u = x^3 - 1$$
- If $f(x) = x\sqrt{1 + x^2}$, you can see that $2x$ is the derivative of $1 + x^2$, so if you make the substitution $u = 1 + x^2$ and write x as $\frac{1}{2}(2x)$, then $f(x) = \frac{1}{2}u^{\frac{1}{2}} \frac{du}{dx}$ where $u = 1 + x^2$ and $\frac{du}{dx} = 2x$.
- If $f(x) = \frac{x+1}{(x^2 + 2x)^3}$, you can see that $2x + 2$ is the derivative of $x^2 + 2x$, so if you make the substitution $u = x^2 + 2x$ and write $x + 1$ as $\frac{1}{2}(2x + 2)$, then $f(x) = \frac{1}{2}u^{-3} \frac{du}{dx}$.
- If $f(x) = x\sqrt{1 - x}$, then you can make the substitution $u = 1 - x$. As $x = 1 - u$, $\frac{du}{dx} = -1$, so:

$$f(x) = -x\sqrt{1 - x} \times (-1)$$

$$= -(1 - u)u^{\frac{1}{2}} \times \frac{du}{dx}$$

$$= -\left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) \times \frac{du}{dx}$$

$$= \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) \times \frac{du}{dx}$$

$$= f(u)\frac{du}{dx}$$

Example 10

Find: **(a)** $\int 3x^2(x^3-1)^4\,dx$ using the substitution $u=x^3-1$

(b) $\int x\sqrt{1+x^2}\,dx$ using the substitution $u=1+x^2$

(c) $\int \frac{x+1}{(x^2+2x)^3}\,dx$ using the substitution $u=x^2+2x$.

Solution

(a) $u=x^3-1, \frac{du}{dx}=3x^2$

$$\begin{aligned}\int 3x^2(x^3-1)^4\,dx &= \int u^4 \times \frac{du}{dx}\,dx\\ &= \int u^4\,du\\ &= \tfrac{1}{5}u^5+C\\ &= \tfrac{1}{5}(x^3-1)^5+C\end{aligned}$$

(b) $u=1+x^2, \frac{du}{dx}=2x$

$$\begin{aligned}\int x\sqrt{1+x^2}\,dx &= \frac{1}{2}\int 2x\sqrt{1+x^2}\,dx\\ &= \frac{1}{2}\int u^{\frac{1}{2}}\times\frac{du}{dx}\,dx\\ &= \frac{1}{2}\int u^{\frac{1}{2}}\,du\\ &= \frac{1}{2}\times\tfrac{2}{3}u^{\frac{3}{2}}+C\\ &= \tfrac{1}{3}(1+x^2)^{\frac{3}{2}}+C\end{aligned}$$

(c) $u=x^2+2x, \frac{du}{dx}=2x+2$

$$\begin{aligned}\int \frac{x+1}{(x^2+2x)^3}\,dx &= \frac{1}{2}\int (2x+2)(x^2+2x)^{-3}\,dx\\ &= \frac{1}{2}\int u^{-3}\times\frac{du}{dx}\,dx\\ &= \frac{1}{2}\int u^{-3}\,du\\ &= \frac{1}{2}\times\left(\frac{1}{-2}\right)u^{-2}+C\\ &= \frac{-1}{4(x^2+2x)^2}+C\end{aligned}$$

A quick way to check your answer is to differentiate it to see that it gives the integrand.

Example 11

Find: **(a)** $\int x\sqrt{1-x}\,dx$ using the substitution $u = 1-x$ **(b)** $\int \frac{t}{\sqrt{1+t}}\,dt$ using the substitution $u = 1+t$

(c) $\int (3x-5)^4\,dx$ using the substitution $u = 3x-5$.

Solution

(a) $u = 1-x,\ \frac{du}{dx} = -1,\ x = 1-u$

$$\begin{aligned}\int x\sqrt{1-x}\,dx &= \int (1-u)u^{\frac{1}{2}} \times \frac{du}{dx}dx \\ &= -\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)(-1)\,dx \\ &= -\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du \qquad \text{(note that } du = (-1)\,dx\text{)} \\ &= -\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + C \\ &= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C\end{aligned}$$

(b) $u = 1+t,\ \frac{du}{dt} = 1,\ t = u-1$

$$\begin{aligned}\int \frac{t}{\sqrt{1+t}}\,dt &= \int \frac{u-1}{\sqrt{u}} \times \frac{du}{dt} \times dt \\ &= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)(1)\,dt \\ &= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du \qquad \text{(note that } du = (1)\,dt\text{)} \\ &= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C \\ &= \frac{2}{3}(1+t)^{\frac{3}{2}} - 2(1+t)^{\frac{1}{2}} + C\end{aligned}$$

(c) $u = 3x-5,\ \frac{du}{dx} = 3$

$$\begin{aligned}\int (3x-5)^4\,dx &= \frac{1}{3}\int 3(3x-5)^4\,dx \\ &= \frac{1}{3}\int u^4 \times \frac{du}{dx}dx \\ &= \frac{1}{3}\int u^4\,du \\ &= \frac{1}{3} \times \frac{1}{5}u^5 + C \\ &= \frac{1}{15}(3x-5)^5 + C\end{aligned}$$