PARALLEL AND PERPENDICULAR LINES IN THREE DIMENSIONS

In two dimensions, two lines can either intersect, be parallel to each other, or be coincident (the same line).

A special case of intersecting is when they intersect at an angle of 90° when they are said to be *perpendicular* to each other. If two or more lines lie on the same plane, they called coplanar.

In two dimensions, all lines lie in the same plane, the x–y plane. All these lines are *coplanar*.

Given line L_1 with equation $y = m_1x + c_1$ and line L_2 with equation $y = m_2x + c_2$, then:

$L_1 \parallel L_2$ if $m_1 = m_2$, $L_1 \perp L_2$ if $m_1 \times m_2 = -1$ and the lines are coincident if $m_1 = m_2$ and $c_1 = c_2$.

Given line L_1 with equation $a_1x + b_1y + c_1 = 0$ and line L_2 with equation $a_2x + b_2y + c_2 = 0$, then:

$L_1 \parallel L_2$ if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $\frac{c_1}{c_2} \neq \frac{a_1}{a_2}$

$L_1 \perp L_2$ if $\frac{a_1}{b_1} \times \frac{a_2}{b_2} = -1$

and the lines are coincident if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $\frac{c_1}{c_2} = \frac{a_1}{a_2}$.

In three dimensions, coplanar lines may intersect, be parallel or be coincident. If they intersect, then they could be perpendicular to each other (orthogonal).

If two lines in three dimensions do not intersect, and are not parallel or coincident, they are called skew lines and are not coplanar. They may still be perpendicular as there may exist a pair of intersecting perpendicular lines, each of which is parallel to one of the given lines.

Given line L_1 with parametric equations $x = a_1 + b_1t$, $y = c_1 + d_1t$, $z = e_1 + f_1t$ and line L_2 with parametric equations $x = a_2 + b_2s$, $y = c_2 + d_2s$, $z = e_2 + f_2s$, then:

- $L_1 \parallel L_2$ if $\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}$
- L_1 and L_2 intersect if a unique value of t and s satisfies the three equations $a_1 + bt = a_2 + b_2s$, $c_1 + d_1t = c_2 + d_2s$, $e_1 + f_1t = e_2 + f_2s$ simultaneously,
- $L_1 \perp L_2$ if $b_1b_2 + d_1d_2 + f_1f_2 = 0$ ($\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$)
- they are skew if they are not parallel, coincident or intersecting.

Note: In three dimensions, coincident lines will be parallel and have a point in common. This means that for L_1 and L_2 above to be coincident, $\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2} = \frac{a_1}{a_2} = \frac{c_1}{c_2} = \frac{e_1}{e_2}$.

Use the parametric form of the equation of a straight line to determine which of the above occur in each case.

Strategy:

1. Determine if the lines are parallel. If yes, determine if they are coincident.
2. Determine if the lines intersect.
3. If they are not parallel or coincident and do not intersect, then they are skew.
4. Determine if they are perpendicular.

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Example 28

For the lines $x = -2 + 2t$, $y = 1 + t$, $z = 4 - 2t$ and $x = 1 + 2s$, $y = 3 + 2s$, $z = 2 - 2s$, determine whether they: **(a)** are parallel, **(b)** intersect, **(c)** are skew, **(d)** are perpendicular.

Solution

(a) Write the ratio of the direction numbers $\frac{2}{2}, \frac{1}{2}, \frac{-2}{-2}$

Since $\frac{2}{2} \neq \frac{1}{2} \neq \frac{-2}{-2}$, the lines are not parallel.

(b) Equate the x, y and z values for each line to obtain 3 equations:

$-2 + 2t = 1 + 2s$ [1]

$1 + t = 3 + 2s$ [2]

$4 - 2t = 2 - 2s$ [3]

Rewrite [1]: $2s = 2t - 3$

Substitute in [2]: $1 + t = 3 + 2t - 3$

$t = 1$

$s = -\frac{1}{2}$

Substitute in [3]: LHS $= 2 - 2 = 0$, RHS $= 2 - 1 = 1$

LHS $\neq$ RHS, so the two lines do not intersect as there is not a unique pair of values for t and s that satisfies all three equations simultaneously.

(c) Hence the lines are skew.

(d) $\underset{\sim}{a} = (2, 1, -2)$, $\underset{\sim}{b} = (2, 2, -2)$ represent the direction vectors of the two lines.

$\underset{\sim}{a} \bullet \underset{\sim}{b} = (2, 1, -2) \bullet (2, 2, -2)$

$= 4 + 2 + 4 = 10 \neq 0$

$\underset{\sim}{a} \bullet \underset{\sim}{b} \neq 0$, hence the lines are not perpendicular.

Example 29

L_1 is $x = 1 + 3t$, $y = 2 + 6t$, $z = 1 - 3t$. L_2 is $x = 2 - 2s$, $y = 1 - 4s$, $z = 2 + 2s$. L_3 is $x = 3 - r$, $y = -1 + r$, $z = -1 + r$.

(a) Show that $L_1 \parallel L_2$.

(b) Show that L_1 and L_3 intersect.

(c) Is L_1 perpendicular to L_3?

(d) What can you say about L_2 and L_3?

Solution

(a) Write the ratio of the direction numbers:

$\frac{3}{-2}, \frac{6}{-4}, \frac{-3}{2}$

In each case, $\frac{t}{s} = -\frac{3}{2}$ so the lines are parallel.

(c) L_1: $\underset{\sim}{a} = (3, 6, -3)$, L_3: $\underset{\sim}{b} = (-1, 1, 1)$. These are the direction vectors for each line.

$\underset{\sim}{a} \bullet \underset{\sim}{b} = (3, 6, -3) \bullet (-1, 1, 1) = -3 + 6 - 3 = 0$

Hence $L_1 \perp L_3$

(d) Since $L_1 \parallel L_2$ and $L_1 \perp L_3$, then $L_2 \perp L_3$.

(b) L_1: $x = 1 + 3t$, $y = 2 + 6t$, $z = 1 - 3t$

L_3: $x = 3 - r$, $y = -1 + r$, $z = -1 + r$

$1 + 3t = 3 - r$ [1]

$2 + 6t = -1 + r$ [2]

$1 - 3t = -1 + r$ [3]

[1] $\Rightarrow$: $r = 2 - 3t$

Substitute in [2]: $2 + 6t = -1 + 2 - 3t$

$9t = -1$

$t = -\frac{1}{9}$ $\quad r = 2\frac{1}{3}$

Substitute in [3]: LHS $= 1 + \frac{3}{9} = 1\frac{1}{3}$,

RHS $= -1 + 2\frac{1}{3} = 1\frac{1}{3}$

LHS = RHS so the two lines intersect.

PARALLEL AND PERPENDICULAR LINES IN THREE DIMENSIONS

Vector equations of a line

Two straight lines will be parallel if their direction vectors are parallel. Similarly, two straight lines will be perpendicular if their direction vectors are perpendicular.

Consider two straight lines, L_1 and L_2, with vector equations $\underset{\sim}{r}_1 = \underset{\sim}{a}_1 + \lambda \underset{\sim}{b}_1$ and $\underset{\sim}{r}_2 = \underset{\sim}{a}_2 + \lambda \underset{\sim}{b}_2$ respectively and with direction vectors $\underset{\sim}{b}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$ respectively.

- If the two lines are parallel, then $\underset{\sim}{b}_1 = k\underset{\sim}{b}_2$, $k \neq 0$, and so $x_1 = kx_2$, $y_1 = ky_2$ and $z_1 = kz_2$

Then, eliminating the constant k, if L_1 and L_2 are parallel, then $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.

- If the two lines are perpendicular, then $\underset{\sim}{b}_1 \bullet \underset{\sim}{b}_2 = 0$ and so $x_1x_2 + y_1y_2 + z_1z_2 = 0$.

Example 30

Line L_1 passes through the points $(1, -2, 4)$ and $(5, 3, -2)$, while line L_2 passes through the points $(3, 8, -2)$ and $(a, -2, 10)$, where $a \in R$.

Find the value(s) of a if: **(a)** L_1 is parallel to L_2. **(b)** L_1 is perpendicular to L_2.

Solution

(a) Find the direction vectors for each of the lines.

Line L_1 passes through the points $(1, -2, 4)$ and $(5, 3, -2)$ and so the direction vector is:

$$\underset{\sim}{b}_1 = (5-1)\underset{\sim}{i} + (3--2)\underset{\sim}{j} + (-2-4)\underset{\sim}{k}$$
$$= 4\underset{\sim}{i} + 5\underset{\sim}{j} - 6\underset{\sim}{k}$$

Line L_2 passes through the points $(3, 8, -2)$ and $(a, -2, 10)$ and so the direction vector is:

$$\underset{\sim}{b}_2 = (a-3)\underset{\sim}{i} + (-2-8)\underset{\sim}{j} + (10--2)\underset{\sim}{k}$$
$$= (a-3)\underset{\sim}{i} - 10\underset{\sim}{j} + 12\underset{\sim}{k}$$

If the two lines are parallel, then $\underset{\sim}{b}_1 = k\underset{\sim}{b}_2$, $k \neq 0$, and so $x_1 = kx_2$, $y_1 = ky_2$ and $z_1 = kz_2$.

If $L_1 \parallel L_2$, then $\underset{\sim}{b}_1 = k\underset{\sim}{b}_2$, $k \neq 0$.

$$4\underset{\sim}{i} + 5\underset{\sim}{j} - 6\underset{\sim}{k} = k\left((a-3)\underset{\sim}{i} - 10\underset{\sim}{j} + 12\underset{\sim}{k}\right) \text{ and } \frac{4}{a-3} = \frac{5}{-10} = \frac{-6}{12}$$

$$\frac{4}{a-3} = -\frac{1}{2}$$
$$a - 3 = -8$$
$$a = -5$$

(b) If the two lines are perpendicular, then $\underset{\sim}{b}_1 \bullet \underset{\sim}{b}_2 = 0$ and so $x_1x_2 + y_1y_2 + z_1z_2 = 0$.

If $L_1 \perp L_2$, then $\underset{\sim}{b}_1 \bullet \underset{\sim}{b}_2 = 0$: $\left(4\underset{\sim}{i} + 5\underset{\sim}{j} - 6\underset{\sim}{k}\right) \bullet \left((a-3)\underset{\sim}{i} - 10\underset{\sim}{j} + 12\underset{\sim}{k}\right) = 0$

$$4 \times (a-3) + 5 \times (-10) + (-6) \times 12 = 0$$
$$4a - 134 = 0$$
$$4a = 134$$
$$a = \frac{67}{2}$$

Example 31

(a) Find the parametric equations of the line through (−2, 1, 2) that is parallel to $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$.

(b) Do either of the the points (1, 2, 3) or (−6, 3, −4) lie on this line?

(c) Find the coordinates of two points that lie on the line in part **(a)**.

Solution

(a) Equation of the line is: $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$$\underset{\sim}{r} = (-2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$$
$$= (-2 + 2\lambda)\underset{\sim}{i} + (1 - \lambda)\underset{\sim}{j} + (2 + 3\lambda)\underset{\sim}{k}$$

The parametric equations of the line are $x = -2 + 2\lambda$, $y = 1 - \lambda$, $z = 2 + 3\lambda$.

(b) Substitute coordinates (1, 2, 3) into one part of the parametric equations to find λ, then see if that value works in the other two parts: $1 = -2 + 2\lambda$ so $\lambda = \frac{3}{2}$.

This gives $y = 1 - \frac{3}{2} = -\frac{1}{2}$ and $z = 2 + \frac{9}{2} = 6\frac{1}{2}$. In the given point, $y = 2$ and $z = 3$ so the point (1, 2, 3) does not lie on the line.
For (−6, 3, −4): $-6 = -2 + 2\lambda$ so $\lambda = -2$.
This gives $y = 1 + 2 = 3$ and $z = 2 - 6 = -4$, which are the corresponding coordinates of the given point. Hence the point (−6, 3, −4) lies on the line.

(c) Give λ two different values: $\lambda = 1$, $x = -2 + 2 = 0$, $y = 1 - 1 = 0$, $z = 2 + 3 = 5$. Hence the point (0, 0, 5) lies on the line.
$\lambda = 2$: $x = 2$, $y = -1$, $z = 8$. Hence the point (2, −1, 8) lies on the line.

Example 32

Find the coordinates of the points where the line $\underset{\sim}{r} = (1 + \lambda)\underset{\sim}{i} + (4 - 2\lambda)\underset{\sim}{j} + (3 + \lambda)\underset{\sim}{k}$ cuts the coordinate planes.

Solution

In the x–y plane, $z = 0$: $3 + \lambda = 0$ so $\lambda = -3$. The point is (−2, 10, 0).

In the x–z plane, $y = 0$: $4 - 2\lambda = 0$ so $\lambda = 2$. The point is (3, 0, 5).

In the y–z plane, $x = 0$: $1 + \lambda = 0$ so $\lambda = -1$. The point is (0, 6, 2).