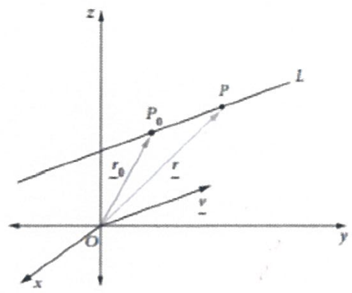
# VECTOR EQUATION OF A LINE




The position of a line in the $x$–$y$ plane is located when a point on the line and its direction, given by either its gradient or the angle of inclination, are known, or by the location of two points through which it passes.

In three-dimensional space, the position of a line is located when you have a point on the line and its direction, this direction given by a vector, or by the location of two points through which it passes.

The diagram to the right shows the line, $L$, passing through the point $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$. $L$ may also be defined by the point $P_0$ and the vector $\overrightarrow{P_0P}$, which determines the direction of the line.

Consider the position vector $\underset{\sim}{v} = l\underset{\sim}{i} + m\underset{\sim}{j} + n\underset{\sim}{k}$ that is parallel to $\overrightarrow{P_0P}$.

Since $\underset{\sim}{v}$ and $\overrightarrow{P_0P}$ are parallel, then $\overrightarrow{P_0P} = \lambda\underset{\sim}{v}$, where $\lambda$ is a real number.

Given that the location of $P$ is specified by position vector $\underset{\sim}{r}$, then $\overrightarrow{OP} = \underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.

Hence $\underset{\sim}{r} = \underset{\sim}{r}_0 + \overrightarrow{P_0P}$

$= \underset{\sim}{r}_0 + \lambda\underset{\sim}{v}$

Thus, the vector equation of the line $L$ is $\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda\underset{\sim}{v}$.

This equation consists of the position vector $\underset{\sim}{r}_0$, a fixed point $P_0$ on $L$, and a fixed vector $\underset{\sim}{v}$ that determines the direction of $L$.

From this equation, you can obtain the parametric equation of the line $L$, namely

$$x = x_0 + a\lambda,\ y = y_0 + b\lambda,\ z = z_0 + c\lambda.$$

They represent the parametric equations of the line $L$ through the point $(x_0, y_0, z_0)$, and parallel to the vector $\underset{\sim}{v} = (a, b, c)$.

Also, the values $a$, $b$, $c$ are called the **direction numbers** of the line as they give its direction.

The vector equation of the line $L$ gives the coordinates of a point for each value of $\lambda$.

For example, if $\lambda = 0$, $\underset{\sim}{r} = \underset{\sim}{r}_0$, then the point is $P_0$.

If $\lambda = 1$, $\underset{\sim}{r} = \underset{\sim}{r}_0 + \underset{\sim}{v}$, then the point is $|\underset{\sim}{v}|$ units to the right of $P_0$. ($\underset{\sim}{v}$ is positive to the right in the diagram.)

If $\lambda = -2$, $\underset{\sim}{r} = \underset{\sim}{r}_0 - 2\underset{\sim}{v}$, then the point is $2|\underset{\sim}{v}|$ units to the left of $P_0$.

Remember, a vector equation generates the infinite set of points that make up the line, each point corresponding to a real value of $\lambda$.

## Example 23

For the straight line with equation $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$ where $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$, find the coordinates of the points on the line for which:

**(a)** $\lambda = 0$ **(b)** $\lambda = 3$ **(c)** $\lambda = 100$ **(d)** $\lambda = -5$.

### Solution

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = \left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) + \lambda\left(\underset{\sim}{i} - \underset{\sim}{j}\right)$

$x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = (1 + \lambda)\underset{\sim}{i} + (1 - \lambda)\underset{\sim}{j} + \underset{\sim}{k}$

Equating components: $x = 1 + \lambda$, $y = 1 - \lambda$, $z = 1$

**(a)** $\lambda = 0$: $x = 1$, $y = 1$, $z = 1$ so the point is $(1, 1, 1)$ **(b)** $\lambda = 3$: $x = 4$, $y = -2$, $z = 1$ so the point is $(3, -2, 1)$

**(c)** $\lambda = 100$: $x = 101$, $y = -99$, $z = 1$ so the point is $(101, -99, 1)$

**(d)** $\lambda = -5$: $x = -4$, $y = 6$, $z = 1$ so the point is $(-4, 6, 1)$.

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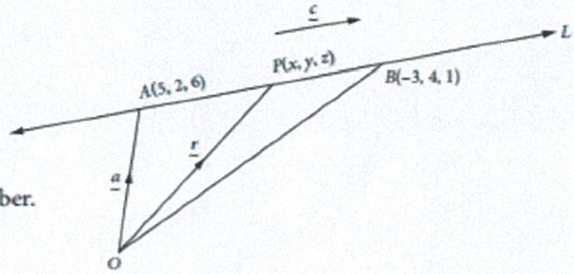
## Example 24

Find the vector equation to the straight line joining the points $A(5, 2, 6)$ and $B(-3, 4, 1)$ and find the coordinates of the point on $AB$ where $\lambda = 2$.

### Solution




$\overrightarrow{OA} = \underset{\sim}{a} = 5\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$

$\overrightarrow{AB} = \underset{\sim}{c} = (-3-5)\underset{\sim}{i} + (4-2)\underset{\sim}{j} + (1-6)\underset{\sim}{k}$

$\underset{\sim}{c} = -8\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}$

The equation of $L$ is given by $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{c}$, $\lambda$ is a real number.

Thus $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = \left(5\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}\right) + \lambda\left(-8\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}\right)$

$= (5 - 8\lambda)\underset{\sim}{i} + (2 + 2\lambda)\underset{\sim}{j} + (6 - 5\lambda)\underset{\sim}{k}$

This gives $x = 5 - 8\lambda$, $y = 2 + 2\lambda$, $z = 6 - 5\lambda$

$\lambda = 2$: $x = -11$, $y = 6$, $z = -4$ so the coordinates of the point are $(-11, 6, -4)$.

## Example 25

Find the vector and parametric equations of the line:

**(a)** through the point $A(-2, 1, 4)$ parallel to the vector $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

**(b)** through the point $A(1, 3, 2)$ parallel to the line through $B(1, 2, 3)$ and $C(3, 4, 1)$.

### Solution

**(a)** $A(-2, 1, 4)$: $\overrightarrow{OA} = \underset{\sim}{a} = -2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$: $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = -2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

Hence the vector equation of the line is $\underset{\sim}{r} = (-2 + 2\lambda)\underset{\sim}{i} + (1 + \lambda)\underset{\sim}{j} + (4 - 2\lambda)\underset{\sim}{k}$.

The parametric equation is $x = -2 + 2\lambda$, $y = 1 + \lambda$, $z = 4 - 2\lambda$. The direction numbers are 2, 1, −2 (the coefficients of $\lambda$).

**(b)** $A(1, 3, 2)$: $\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{b} = \overrightarrow{BC} = (3-1)\underset{\sim}{i} + (4-2)\underset{\sim}{j} + (1-3)\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

Hence the vector equation of the line is $\underset{\sim}{r} = (1 + 2\lambda)\underset{\sim}{i} + (3 + 2\lambda)\underset{\sim}{j} + (2 - 2\lambda)\underset{\sim}{k}$.

The parametric equation is $x = 1 + 2\lambda$, $y = 3 + 2\lambda$, $z = 2 - 2\lambda$.

## Example 26

Given $P_1(2, 1, 2)$ and $P_2(-1, 3, 3)$. With $P_1$ as the fixed point, find:

**(a)** the vector equation of the line $L$ that passes through $P_1$ and $P_2$

**(b)** the parametric equation of the line $L$ that passes through $P_1$ and $P_2$.

With $P_2$ as the fixed point, find:

**(c)** the vector equation of the line $L$ that passes through $P_1$ and $P_2$

**(d)** the parametric equation of the line $L$ that passes through $P_1$ and $P_2$.

**(e)** Discuss your answers to parts **(b)** and **(d)**.

## Solution

**(a)** $\underset{\sim}{v} = \overrightarrow{P_1P_2} = (-1-2)\underset{\sim}{i} + (3-1)\underset{\sim}{j} + (3-2)\underset{\sim}{k}$

$= -3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$P_1(2, 1, 2)$: $\underset{\sim}{r_0} = \overrightarrow{OP_1} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$L$ is parallel to $\overrightarrow{P_1P_2}$, so the vector equation of $L$ is $\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda\underset{\sim}{v}$, where $\underset{\sim}{r} = (x, y, z)$.

Hence the equation of $L$ is $\underset{\sim}{r} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} + \lambda\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

$\underset{\sim}{r} = (2-3\lambda)\underset{\sim}{i} + (1+2\lambda)\underset{\sim}{j} + (2+\lambda)\underset{\sim}{k}$

**(b)** The parametric equations of $L$ are: $x = 2 - 3\lambda$, $y = 1 + 2\lambda$, $z = 2 + \lambda$.

**(c)** $P_2(-1, 3, 3)$: $\underset{\sim}{r_0} = \overrightarrow{OP_2} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{v} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$: $\underset{\sim}{r} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k} + \lambda\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

$\underset{\sim}{r} = (-1-3\lambda)\underset{\sim}{i} + (3+2\lambda)\underset{\sim}{j} + (3+\lambda)\underset{\sim}{k}$

**(d)** The parametric equations of $L$ are: $x = -1 - 3\lambda$, $y = 3 + 2\lambda$, $z = 3 + \lambda$.

**(e)** The equations of the line $L$ are different when they have the different fixed points. To show that each equation represents the same line, find values of $\lambda$ that generate the points $P_1$ and $P_2$ for each equation. This is most easily done from the parametric equation. Obviously, $\lambda = 0$ will generate the fixed point in each case.

For part **(b)**: $x = 2 - 3\lambda$, $y = 1 + 2\lambda$, $z = 2 + \lambda$, for $P_2$: $2 - 3\lambda = -1$, $\lambda = 1$.
Substitute for $y$ and $z$: $y = 1 + 2 = 3$. $z = 2 + 1 = 3$. Hence the point is $(-1, 3, 3)$, which is $P_2$.

For part **(d)**: $x = -1 - 3\lambda$, $y = 3 + 2\lambda$, $z = 3 + \lambda$, for $P_1$: $-1 - 3\lambda = 2$, $\lambda = -1$.
Substitute for $y$ and $z$: $y = 3 - 2 = 1$. $z = 3 - 1 = 2$. Hence the point is $(2, 1, 2)$, which is $P_1$.
Thus, both sets of equations represent the line $L$ through the points $P_1$ and $P_2$.

If $A$, $B$ and $R$ are points on a straight line through $A$ and $B$ such that $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{AB}$ and $\underset{\sim}{r} = \overrightarrow{OR}$, then the vector equation of the line $AB$ is given by $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$, where $\lambda$ is a parameter.

When starting with the coordinates of two points to form the vector equation of a line, the final equation depends on which point you fix to obtain $\underset{\sim}{a}$. Both choices will lead to a correct vector equation defining the line.

Hence the vector (or parametric) equation of a line through two points is not unique.

# VECTOR EQUATION OF A LINE

## Vector equations of a line in two dimensions

**Gradient intercept form**

Consider the equation of the line given by $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$, where $\underset{\sim}{r} = (x, y)$, $\underset{\sim}{a} = (x_0, y_0)$ and $\underset{\sim}{b} = (x_1, y_1)$.

The vector equation becomes $\underset{\sim}{r} = (x_0 + x_1\lambda)\underset{\sim}{i} + (y_0 + y_1\lambda)\underset{\sim}{j}$.

The parametric equations are $x = x_0 + x_1\lambda$, $y = y_0 + y_1\lambda$.

Rearranging the first equation gives $\lambda = \frac{x - x_0}{x_1}$

Substitute in the second equation for $y$: $y = y_0 + \frac{y_1(x - x_0)}{x_1}$

$$y = y_0 + \frac{y_1}{x_1}x - \frac{x_0 y_1}{x_1}$$

$$y = \frac{y_1}{x_1}x + \frac{x_1 y_0 - x_0 y_1}{x_1}$$

Comparing this answer with the gradient–intercept form of the straight line, $y = mx + c$, gives $m = \frac{y_1}{x_1}$ and $c = \frac{x_1 y_0 - x_0 y_1}{x_1}$.

Thus, the gradient of the line is $\frac{y_1}{x_1}$ and the $y$-intercept is $\frac{x_1 y_0 - x_0 y_1}{x_1}$.

The vector $\underset{\sim}{b} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ is a vector parallel to the line with gradient $\frac{y_1}{x_1}$. You would expect the gradients of the line and the vector to be the same.

**Two-point form**

In two-dimensional coordinate geometry, the equation of the line through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$, provided that $x_2 \neq x_1$.

The vector equation of the line $AB$ is $x\underset{\sim}{i} + y\underset{\sim}{j} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + \lambda\left((x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j}\right)$ as $(x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j}$ is a vector parallel to $\overrightarrow{AB}$.

Thus $x = x_1 + \lambda(x_2 - x_1)$

$\quad y = y_1 + \lambda(y_2 - y_1)$

Solving both of these equations for $\lambda$ gives $\frac{x - x_1}{x_2 - x_1} = \lambda = \frac{y - y_1}{y_2 - y_1}$ [1]

This means that $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$, which is the two-point form of the Cartesian equation of a line.

If $\lambda = 0$, then equation [1] gives $x = x_1$, $y = y_1$, or $(x_1, y_1)$, the coordinates of the fixed point for the vector equation.

### Example 27

Find the vector and Cartesian equations of the line through the point $A(2, -1)$ parallel to the vector $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j}$.

### Solution

$\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j}$

Vector equation: $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$: $\underset{\sim}{r} = 2\underset{\sim}{i} - \underset{\sim}{j} + \lambda\left(\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

$$\underset{\sim}{r} = (2 + \lambda)\underset{\sim}{i} + (-1 + 2\lambda)\underset{\sim}{j}$$

Parametric equations: $x = 2 + \lambda$, $y = -1 + 2\lambda$

$\lambda = x - 2$ so $y = -1 + 2(x - 2)$

$y = 2x - 5$ is the Cartesian equation of the line.