MATHEMATICAL INDUCTION INVOLVING SERIES

1 If $S(n)$ is the statement that $n + 2n + 3n + \ldots + n^2 = \frac{n^2(n+1)}{2}$, then $S(5)$ represents the statement:

A $1+2+3+\ldots+5=\frac{5\times 6}{2}$

B $1+2+3+\ldots+25=\frac{25\times 26}{2}$

C $1+2+3+\ldots+25=\frac{25\times 6}{2}$

$\boxed{\text{D}}$ $5+10+15+20+25=\frac{25\times 6}{2}$

$n=5$

Prove each of the following by induction for all positive integers n.

3 $1+2+4+\ldots+2^{n-1}=2^n-1$

Step 1: For $n=2$ $\quad 1+2^{2-1}=2^2-1=3$ True.

Step 2: We assume it's true for $n=k$, i.e $1+2+4+\ldots+2^{k-1}=2^k-1$

In that case: $1+2+4+\ldots+2^{k-1}+2^k=(2^k-1)+2^k$

$=2\times 2^k-1$

$=2^{k+1}-1$ also true for $n=k+1$

Step 3: – the statement is true for $n=2$

– it's true for $n=k+1$ if it's true for $n=k$

– $\therefore$ it's true for any $n \geq 2$

4 $2+5+8+\ldots+(3n-1)=\dfrac{n(3n+1)}{2}$

Step 1: for $n=2$ $\quad \underbrace{2+(3\times 2-1)}_{=7} = \dfrac{2(3\times 2+1)}{2} = 7$ true.

Step 2 we assume it's true for $n=k$, i.e. $2+5+8+\ldots+(3k-1)=\dfrac{k(3k+1)}{2}$

In that case $2+5+8+\ldots+(3k-1)+\underbrace{(3(k+1)-1)}_{=3k+2} = \dfrac{k(3k+1)}{2}+(3k+2)$

$$= \frac{k(3k+1)+6k+4}{2}$$

$$= \frac{3k^2+7k+4}{2} = \frac{(k+1)(3k+4)}{2}$$

$$= \frac{(k+1)(3(k+1)+1)}{2}$$

Step 3: – the statement is true for $n=2$

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– it's true for $n=k+1$ if it's true for $n=k$

$\therefore$ it's true for any $n \geq 2$

7 $1+r+r^2+r^3+\ldots+r^{n-1}=\dfrac{1-r^n}{1-r}$

Step 1: $n=2$ $\quad 1+r^{2-1}=1+r$ whereas $\dfrac{1-r^2}{1-r}=\dfrac{(1-r)(1+r)}{(1-r)}=1+r$

So true for $n=2$

Step 2 we assume it holds for $n=k$

So for $n=k+1$ $\quad 1+r+r^2+\ldots+r^{k-1}+r^k=\dfrac{1-r^k}{1-r}+r^k$

$$=\frac{1-r^k+r^k(1-r)}{1-r}=\frac{1-r^{k+1}}{1-r}$$

Step 3 – it's true for $n=2$

– it's true for $n=k+1$ if it's true for $n=k$

$\therefore$ it's true for any $n \geqslant 2$.

8 $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

Step 1 $n=2$ $1 \times 2 + 2(2+1) = 8$

whereas $\dfrac{2(2+1)(2+2)}{3} = 8$ so it's true for $n=2$

Step 2 we assume it's true for $n=k$

i.e. $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$

In that case:

$1 * 2 + 2 \times 3 + \cdots + k(k+1) + (k+1)(k+2) = \dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2)$

$\_\_\_\_\_\_\_\_ = \dfrac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$

$\_\_\_\_\_\_\_\_ = \dfrac{(k+1)(k+2)(k+3)}{3}$

So if it's true for $n=k$, it's also true for $n=k+1$

Step 3. the statement is true for $n=2$

it's true for $n=k+1$ if it's true for $n=k$

$\therefore$ it's true for any $n \geq 2$, by induction.

11 $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Step 1 $n=2$ $\frac{1}{1\times2}+\frac{1}{2(2+1)}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$

whereas $\frac{2}{2+1}=\frac{2}{3}$ so it's true for $n=2$

Step 2 we assume it's true for $n=k$.

So $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

In that case:

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$\rule{6cm}{0.4pt}=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$\rule{4cm}{0.4pt}=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$\rule{4cm}{0.4pt}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

So it's then also true for $k+1$

Step 3 :- we demonstrated the statement is true for $n=2$
- if it's true for $n=k$, then it's also true for $n=k+1$

$\therefore$ it's true for any $n \geqslant 2$, by induction.

15 **(a)** $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$. Hence find $\lim_{n\to\infty}\left(\frac{1^3+2^3+3^3+\ldots+n^3}{n^4}\right)$

(b) Hence show that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1+2+3+\ldots+n)^2$.

a) Step 1 for $n=2$ $\quad 1^3 + 2^3 = \frac{2^2(2+1)^2}{4} = 9$ true.

Step 2 we assume it's true for $n=k$

In that case:

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$= \frac{(k+1)^2(k^2+4(k+1))}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

So if the statement is true for $n=k$, it's also true for $(k+1)$.

Step 3 it's true for $n=2$; if it's true for k, then it's true for $k+1$

So it's true for any $n \geq 2$

Hence $\lim_{n\to+\infty}\left(\frac{1^3+2^3+3^3+\ldots+n^3}{n^4}\right) = \lim_{n\to+\infty}\left(\frac{n^2(n+1)^2}{4n^4}\right)$

$$= \lim_{n\to+\infty}\left[\frac{(n+1)^2}{4n^2}\right] = \frac{1}{4}$$

b) We demonstrated before that $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

So $(1+2+3+\ldots+n)^2 = \frac{n^2(n+1)^2}{4}$

Hence: $1^3+2^3+3^3+\ldots+n^3 = (1+2+3+\ldots+n)^2$

16 $(n+1)+(n+2)+\ldots+2n=\dfrac{n(3n+1)}{2}$

Step 1 $\quad n=2 \quad (2+1)+4=7$

whereas $\dfrac{2(3\times2+1)}{2}=7$ too, so it's true for $n=2$

Step 2 we assume it's true for $n=k$

In that case: ~~(k+1)+(k+2)+...+2(k+1)=~~

$(k+2)+(k+3)+(k+4)+\ldots+2(k+1)=$

$(k+1)+(k+2)+\ldots+2k-(k+1)+2(k+1)+(2k+1)$

$\_\_\_\_ = \dfrac{k(3k+1)}{2}+3k+2$

$\_\_\_\_ = \dfrac{k(3k+1)+6k+4}{2}$

$\_\_\_\_ = \dfrac{3k^2+7k+4}{2}$

$\_\_\_\_ = \dfrac{(k+1)(3k+4)}{2} = \dfrac{(k+1)[3(k+1)+1]}{2}$

So it's then also true for $k+1$

Step 3 : we have shown the statement is true for $n=2$

If it's true for $n=k$ then it's also true for $n=k+1$

$\therefore$ the statement is true for any $n\geq 2$

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17 $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$

Step 1 for $n=1$ $\quad 1 \times 1! = 1$

whereas $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$ so true for $n=1$

Step 2 we assume it's true for $n = k$

In that case: $1 \times 1! + 2 \times 2! + \ldots + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$

$$= (k+1)!\,[k+1+1] - 1$$

$$= (k+1)! \times (k+2) - 1$$

$$= (k+2)! - 1$$

So the statement is then also true for $k+1$.

Step 3 : - we proved that the statement is true for $n=1$

- we showed that it's true for $n = k+1$ if it's true for $n = k$

Therefore, it's true for any $n \geq 1$, by induction.