GEOMETRIC SEQUENCES

A **geometric sequence** is a sequence of terms in which each term after the first term is formed by multiplying the preceding term by a constant number called the **common ratio**.

a is the first term, *r* is the common ratio and *n* is the number of terms.

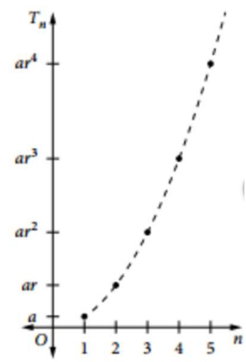
The sequence $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$ is a geometric sequence with *n* terms, where *a* and *r* are real numbers and *n* is a positive integer.

$$T_n = rT_{n-1}$$

This definition is called a recursive definition because it defines a term of the sequence in terms of the preceding term. Rewriting this definition gives $r = \frac{T_n}{T_{n-1}}$ and this result is used to identify whether a sequence is geometric. If there is a constant ratio between successive terms then the sequence is geometric.
A geometric sequence may be defined by its *n*th term, given by the following equation.

$$T_n = ar^{n-1}$$

Graphically, T_n as a function of *n* is exponential in shape, consisting of a series of points for integer values of *n*, $n > 0$, which follow the exponential function given by $T_n = ar^{n-1}$, where the independent variable is *n* and the dependent variable is T_n.



Some examples of geometric sequences are:

$2, 6, 18, 54, \ldots$	$a = 2, r = 3$
$3, -15, 75, -375, \ldots$	$a = 3, r = -5$
$18, 9, 4\frac{1}{2}, 2\frac{1}{4}, \ldots$	$a = 18, r = \frac{1}{2}$
$3, 3\sqrt{2}, 6, 6\sqrt{2}, \ldots$	$a = 3, r = \sqrt{2}$
$x, x^2, x^3, x^4, \ldots$	$a = x, r = x$

Example 19

By finding the ratio between successive terms, identify the sequences that are geometric.

(a) $2, 4, 8, 16, \ldots$ **(b)** $3, 6, 24, 192, \ldots$ **(c)** $48, 24, 12, 6, \ldots$

(d) $x, x^2+1, x^3+2, x^4+3, \ldots$ **(e)** $\frac{1}{\sqrt{2}-1}, 3+2\sqrt{2}, 7+5\sqrt{2}, 17+12\sqrt{2}, \ldots$

Solution

(a) $\frac{4}{2} = \frac{8}{4} = \frac{16}{8} = 2$

$r = 2$, a constant ratio. The sequence is geometric.

(b) $\frac{6}{3} = 2, \frac{24}{6} = 4$

The ratio is not constant so the sequence is not geometric.

(c) $\frac{24}{48} = \frac{12}{24} = \frac{6}{12} = \frac{1}{2}$

$r = \frac{1}{2}$, a constant ratio. The sequence is geometric.

(d) $\frac{x^2+1}{x} \neq \frac{x^2+2}{x^2+1}$

The ratio is not constant so the sequence is not geometric.

(e) $\frac{3+2\sqrt{2}}{\left(\frac{1}{\sqrt{2}-1}\right)} = (3+\sqrt{2})(\sqrt{2}-1) = \sqrt{2}+1$

$\frac{7+5\sqrt{2}}{3+2\sqrt{2}} = \frac{(7+5\sqrt{2})(3-2\sqrt{2})}{9-8} = 21 - 14\sqrt{2} + 15\sqrt{2} - 20 = \sqrt{2}+1$

$\frac{17+12\sqrt{2}}{7+5\sqrt{2}} = \frac{(17+12\sqrt{2})(7-5\sqrt{2})}{49-50} = -(119 - 85\sqrt{2} + 84\sqrt{2} - 120) = \sqrt{2}+1$

$r = \sqrt{2}+1$, a constant ratio. The sequence is geometric.

Example 20

For the geometric sequence 2, 6, 18, 54, … find:

(a) the value of a
(b) the value of r
(c) the expression for T_n
(d) the 10th term
(e) the value of k if $T_k = 4374$.

Solution

(a) $a = 2$

(b) $r = \frac{6}{2} = 3$

(c) $T_n = ar^{n-1} = 2 \times 3^{n-1}$

(d)
$$n = 10$$
$$T_{10} = 2 \times 3^9$$
$$= 39\,366$$

(e)
$$2 \times 3^{k-1} = 4374$$
$$3^{k-1} = 2187 = 3^7$$
$$k - 1 = 7$$
$$k = 8$$

Example 21

If $T_3 = 12$ and $T_6 = 96$ are two terms of a geometric sequence, find the values of a and r and use them to write down the first three terms of the sequence.

Solution

$$T_3 = 12 = ar^2 \quad [1]$$
$$T_6 = 96 = ar^5 \quad [2]$$

[2] ÷ [1]: $\frac{ar^5}{ar^2} = \frac{96}{12}$

$$r^3 = 8$$
$$r = 2$$
$$a \times 2^2 = 12$$
$$a = 3$$

The geometric sequence begins: 3, 6, 12.