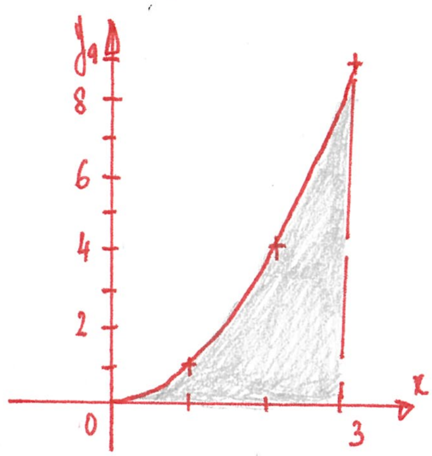
VOLUMES OF SOLIDS OF REVOLUTION

1 Find the volume of the solid of revolution formed by rotating about the x-axis the arc of the parabola $y = x^2$ between $x = 0$ and $x = 3$.



$$V = \int_0^3 \pi (x^2)^2 dx = \int_0^3 \pi x^4 dx$$

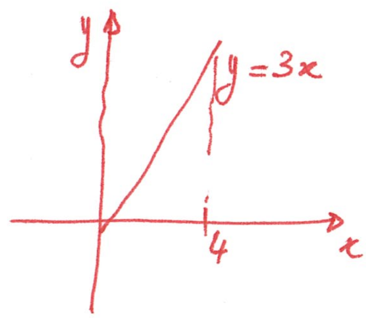
$$V = \pi \int_0^3 x^4 dx = \pi \left[\frac{x^5}{5}\right]_0^3$$

$$V = \pi \left[\frac{3^5}{5} - \frac{0^5}{5}\right] = \frac{\pi 3^5}{5}$$

$$V = \frac{243\pi}{5} \text{ units}^3$$

3 A cone is formed by rotating about the x-axis a segment of the line $y = 3x$ between $x = 0$ and $x = 4$. The definite integral used to calculate the volume of this solid is:

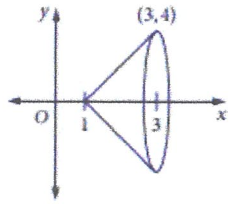
A $\int_0^4 9x^2\,dx$ **B** $\pi\int_0^4 3x^2\,dx$ **C** $\int_0^4 3x^2\,dx$ $\boxed{\textbf{D}}$ $\pi\int_0^4 9x^2\,dx$



$$V = \int_0^4 \pi (3x)^2 dx = 9\pi \int_0^4 x^2 dx$$

VOLUMES OF SOLIDS OF REVOLUTION

4 **(a)** Find the equation of the line passing through the points $(1,0)$ and $(3,4)$.
(b) A cone is formed by rotating about the x-axis the segment of the line joining the points $(1,0)$ and $(3,4)$. Calculate the volume of the cone.



a) $m = \dfrac{4-0}{3-1} = 2$ $\quad y - 0 = 2(x-1)$

So $y = 2x - 2$

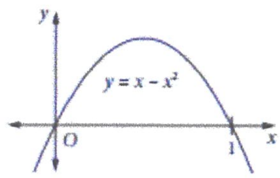
b) $V = \displaystyle\int_1^3 \pi\,(2x-2)^2\,dx = \pi \int_1^3 (4x^2 - 8x + 4)\,dx$

$V = \pi \left[\dfrac{4x^3}{3} - \dfrac{8x^2}{2} + 4x\right]_1^3 = \pi \left[\dfrac{4x^3}{3} - 4x^2 + 4x\right]_1^3$

$V = 4\pi \left[\dfrac{x^3}{3} - x^2 + x\right]_1^3 = 4\pi \left[\left(\dfrac{3^3}{3} - 3^2 + 3\right) - \left(\dfrac{1^3}{3} - 1^2 + 1\right)\right]$

$V = 4\pi \left[9 - 9 + 3 - \left(\dfrac{1}{3}\right)\right] = 4\pi\left(3 - \dfrac{1}{3}\right) = 4\pi \times \dfrac{8}{3} = \dfrac{32\pi}{3}$ units^3

6 The region bounded by the parabola $y = x - x^2$ and the x-axis is rotated about the x-axis. Find the volume of the solid formed.



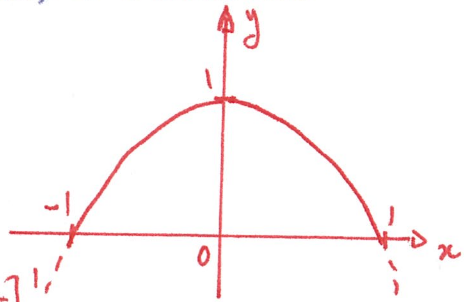
$V = \displaystyle\int_0^1 \pi\,(x - x^2)^2\,dx = \pi \int_0^1 (x^2 - 2x^3 + x^4)\,dx$

$V = \pi \left[\dfrac{x^3}{3} - \dfrac{2x^4}{4} + \dfrac{x^5}{5}\right]_0^1 = \pi \left[\dfrac{1^3}{3} - \dfrac{1^4}{2} + \dfrac{1^5}{5}\right]$

$V = \pi \left[\dfrac{1}{3} - \dfrac{1}{2} + \dfrac{1}{5}\right] = \dfrac{\pi}{30}$ units^3

VOLUMES OF SOLIDS OF REVOLUTION

7 Find the volume of the solid formed when the region bounded by the parabola $y = 1 - x^2$ and the x-axis is rotated about: **(a)** the x-axis **(b)** the y-axis.



a) $V = \int_{-1}^{1} \pi \left(1 - x^2\right)^2 dx$

$V = \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4\right) dx = \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1}$

$V = \pi \left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left((-1) - \frac{2 \times (-1)^3}{3} + \frac{(-1)^5}{5}\right)\right]$

$V = \pi \left[\frac{8}{15} - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right] = \pi \left[\frac{8}{15} - \left(\frac{-8}{15}\right)\right] = \frac{16\pi}{15}$ units^3

b) $V = \int_{0}^{1} \pi (x)^2 \, dy$ But $y = 1 - x^2$ so $x^2 = 1 - y$

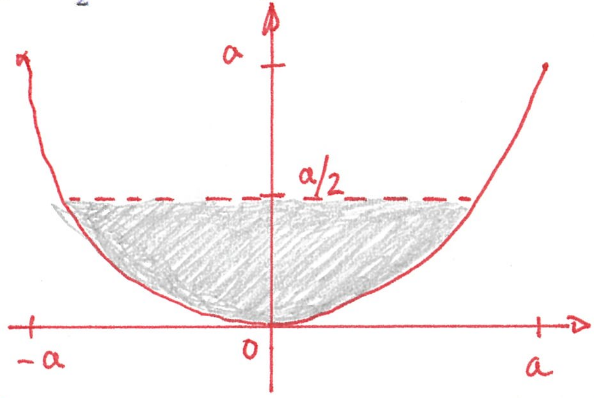
$V = \int_{0}^{1} \pi (1 - y) \, dy$

$V = \pi \left[y - \frac{y^2}{2}\right]_{0}^{1}$

$V = \pi \left[1 - \frac{1^2}{2}\right] = \frac{\pi}{2}$ units^3

13 A hemispherical bowl of radius a units is filled with water to a depth of $\frac{a}{2}$ units. Use integration to find the volume of the water.

Centre $(0, a)$ Radius a



So $x^2 + (y-a)^2 = a^2$

$x^2 + y^2 - 2ay + a^2 = a^2$

$x^2 + y^2 - 2ay = 0.$

When $y = a/2$ then $x^2 + \left(\frac{a}{2}\right)^2 - 2a\left(\frac{a}{2}\right) = 0$ so $x^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$

$x = \frac{\sqrt{3}a}{2}$

$$V = \int_0^{a/2} \pi (x)^2 \, dy = \int_0^{a/2} \pi \left(2ay - y^2\right) dy$$

$$V = \pi \left[\frac{2ay^2}{2} - \frac{y^3}{3}\right]_0^{a/2}$$

$$V = \pi \left[ay^2 - \frac{y^3}{3}\right]_0^{a/2}$$

$$V = \pi \left[a\left(\frac{a}{2}\right)^2 - \frac{(a/2)^3}{3}\right]$$

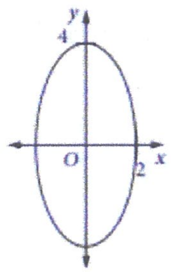
$$V = \pi a^3 \left[\frac{1}{4} - \frac{1}{24}\right]$$

$$V = \frac{5\pi a^3}{24} \text{ units}^3$$

VOLUMES OF SOLIDS OF REVOLUTION

18 Find the volume of the solid formed when the ellipse $4x^2 + y^2 = 16$ is rotated about:

(a) the x-axis **(b)** the y-axis.



a) $V = \int_{-2}^{2} \pi (y)^2 \, dx = \int_{-2}^{2} \pi (16 - 4x^2) \, dx$

$V = 4\pi \int_{-2}^{2} (4 - x^2) \, dx = 4\pi \left[4x - \frac{x^3}{3} \right]_{-2}^{2}$

$V = 4\pi \left[\left(4\times 2 - \frac{2^3}{3} \right) - \left(4\times(-2) - \frac{(-2)^3}{3} \right) \right] = 4\pi \left[\frac{16}{3} - \left(-\frac{16}{3} \right) \right]$

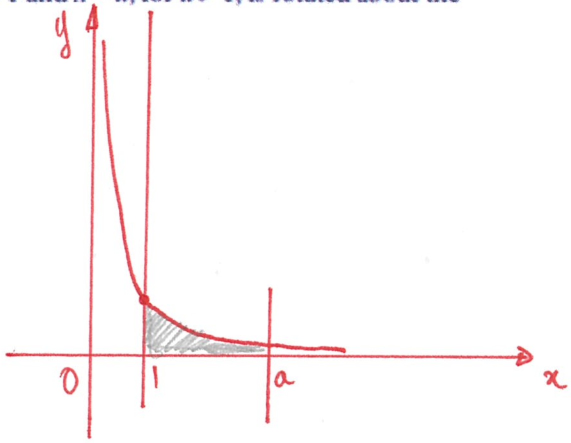
$V = \frac{128\pi}{3}$ units^3

b) $V = \int_{-4}^{4} \pi (x)^2 \, dy = \int_{-4}^{4} \pi x^2 \, dy = \pi \int_{-4}^{4} \left(\frac{16 - y^2}{4} \right) dy$

$V = \frac{\pi}{4} \left[16y - \frac{y^3}{3} \right]_{-4}^{4} = \frac{\pi}{4} \left[\left(16\times 4 - \frac{4^3}{3} \right) - \left(16\times(-4) - \frac{(-4)^3}{3} \right) \right]$

$V = \frac{\pi}{4} \left[\frac{128}{3} - \left(-\frac{128}{3} \right) \right] = \frac{64\pi}{3}$ units^3

20 The region bounded by the curve $xy = 1$, the x-axis and the lines $x = 1$ and $x = a$, for $a > 1$, is rotated about the x-axis. Find V, the volume generated. Hence find $\lim\limits_{a \to \infty} V$.



$$V = \int_1^a \pi (y)^2 \, dx = \int_1^a \pi y^2 \, dx$$

$$V = \int_1^a \pi \left(\frac{1}{x^2}\right) dx = \pi \int_1^a \frac{1}{x^2} \, dx$$

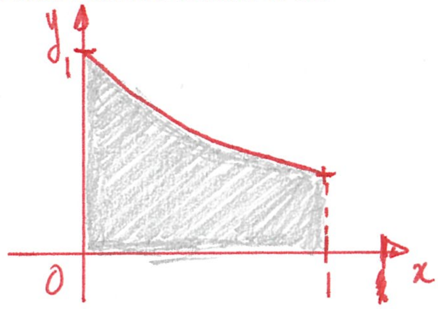
$$V = \pi \int_1^a x^{-2} \, dx = \pi \left[\frac{x^{-1}}{-1}\right]_1^a = \pi \left[-\frac{1}{x}\right]_1^a$$

$$V = \pi \left[-\frac{1}{a} - \left(-\frac{1}{1}\right)\right] = \pi \left[1 - \frac{1}{a}\right] \text{ units}^3$$

$$\text{So } \lim_{a \to +\infty} V = \lim_{a \to +\infty} \pi \left[1 - \frac{1}{a}\right] = \pi \text{ units}^3$$

VOLUMES OF SOLIDS OF REVOLUTION

25 The area under the curve $y = e^{-x}$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Find the volume of the solid of revolution.



$$V = \int_0^1 \pi (y)^2 \, dx = \int_0^1 \pi y^2 \, dx$$

$$V = \pi \int_0^1 (e^{-x})^2 \, dx = \pi \int_0^1 e^{-2x} \, dx$$

$$V = \pi \left[\frac{e^{-2x}}{-2} \right]_0^1 = \pi \left[\frac{e^{-2}}{-2} - \frac{e^0}{-2} \right] = \pi \left[\frac{1}{2} - \frac{1}{2e^2} \right]$$

$$V = \frac{\pi}{2} \left[1 - \frac{1}{e^2} \right] = \frac{\pi}{2} \left(\frac{e^2 - 1}{e^2} \right) \text{ units}^3$$

27 Find the volume generated when the curve $y = e^{-0.5x}$, $-2 \le x \le 2$, is rotated about the x-axis.

$$V = \int_{-2}^{2} \pi \left(e^{-0.5x} \right)^2 \, dx = \pi \int_{-2}^{2} e^{-x} \, dx$$

$$V = \pi \left[\frac{e^{-x}}{-1} \right]_{-2}^{2} = \pi \left[\frac{e^{-2}}{-1} - \frac{e^2}{-1} \right] = \pi \left[e^2 - \frac{1}{e^2} \right]$$

$$\text{So } V = \pi \left(\frac{e^4 - 1}{e^2} \right) \text{ units}^3$$

29 **(a)** Find the area of the region bounded by the curve $y = e^{-x}$, the coordinate axes and the line $x = a$, $a > 0$.
(b) Find the limit of this area as $a \to \infty$.
(c) Find the volume of the solid generated by rotating the region in **(a)** about the x-axis and find the limit of this volume as $a \to \infty$.

a) $\text{Area} = \int_0^a e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^a = \left[\frac{e^{-a}}{-1} - \frac{e^0}{-1}\right]$

$\text{Area} = 1 - \frac{1}{e^a}$

b) $\lim_{a \to +\infty}\left(1 - \frac{1}{e^a}\right) = 1$ as $\lim_{a \to +\infty} e^a = +\infty$

c) $V = \int_0^a \pi y^2\,dx = \pi \int_0^a e^{-2x}\,dx = \pi\left[\frac{e^{-2x}}{-2}\right]_0^a$

$V = \pi\left(\frac{e^{-2a}}{-2} - \frac{e^0}{-2}\right) = \frac{\pi}{2}\left(1 - e^{-2a}\right)$

$V = \frac{\pi}{2}\left(1 - \frac{1}{e^{2a}}\right)$

$\lim_{a \to +\infty} V = \lim_{a \to +\infty} \frac{\pi}{2}\left(1 - \frac{1}{e^{2a}}\right) = \frac{\pi}{2}$

as $\lim_{a \to +\infty} \frac{1}{e^{2a}} = 0$

$\left(\text{as } \lim_{a \to +\infty} e^{2a} = +\infty\right)$

31 Find the volume of the solid generated by rotating about the x-axis the area beneath the curve $y = \frac{1}{\sqrt{x-2}}$ between $x = 6$ and $x = 11$.

$$V = \int_6^{11} \pi y^2 \, dx \qquad y = \frac{1}{\sqrt{x-2}} \text{ so } y^2 = \frac{1}{x-2}$$

$$V = \int_6^{11} \frac{\pi}{x-2} \, dx$$

$$V = \pi \int_6^{11} \frac{1}{x-2} \, dx = \pi \left[\ln(x-2) \right]_6^{11}$$

$$V = \pi \left[\ln 9 - \ln 4 \right]$$

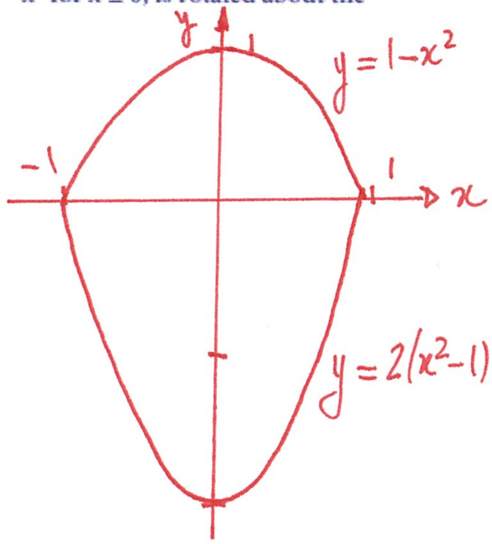
$$V = \pi \ln\left(\frac{9}{4}\right) \text{ units}^3$$

37 (a) Sketch the region bounded by the curves $y = 2(x^2 - 1)$ and $y = 1 - x^2$.

(b) Calculate the area of the shaded region.

(c) The region bounded by the y-axis and the curves $y = 2(x^2 - 1)$ and $y = 1 - x^2$ for $x \geq 0$, is rotated about the y-axis. Calculate the volume of the solid of revolution generated.

a)



b) $\text{Area} = \int_{-1}^{1} (1 - x^2)\,dx - \int_{-1}^{1} 2(x^2 - 1)\,dx$

$\text{Area} = \int_{-1}^{1} [1 - x^2 - 2x^2 + 2]\,dx$

$\text{Area} = \int_{-1}^{1} (3 - 3x^2)\,dx = 3\int_{-1}^{1} (1 - x^2)\,dx$

$\text{Area} = 3\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 3\left[\left(1 - \frac{1}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right)\right]$

$\text{Area} = 3\left(\frac{2}{3} + \frac{2}{3}\right) = 4 \text{ units}^2$

c) $V = \int_{-2}^{1} \pi x^2\,dy = \int_{-2}^{0} \pi x^2\,dy + \int_{0}^{1} \pi x^2\,dy$

$V = \pi\int_{-2}^{0} x^2\,dy + \pi\int_{0}^{1} x^2\,dy$

$y = 2(x^2 - 1)$ so $x^2 = 1 + \frac{y}{2}$

$y = 1 - x^2$ so $x^2 = 1 - y$

$V = \pi\int_{-2}^{0} \left(1 + \frac{y}{2}\right) dy + \pi\int_{0}^{1} (1 - y)\,dy$

$V = \pi\left[y + \frac{y^2}{4}\right]_{-2}^{0} + \pi\left[y - \frac{y^2}{2}\right]_{0}^{1}$

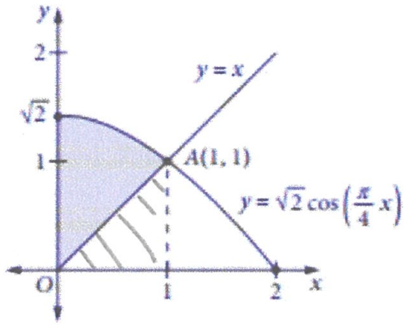
$V = \pi\left[-\left(-2 + \frac{(-2)^2}{4}\right)\right] + \pi\left[1 - \frac{1^2}{2}\right] = \pi[1] + \pi \times \frac{1}{2}$

$V = \frac{3\pi}{2} \text{ units}^3$

VOLUMES OF SOLIDS OF REVOLUTION

38 The curve $y = \sqrt{2}\cos\left(\frac{\pi}{4}x\right)$ meets the line $y = x$ at the point $A(1, 1)$, as shown in the diagram.

(a) Find the exact value of the shaded area.
(b) The shaded area is rotated about the x-axis. Calculate the volume of the solid of revolution formed.
(c) The shaded area is rotated about the y-axis. Write the integral for this volume.
~~**(d)** By using a combination of exact integration and the trapezoidal rule, as appropriate, calculate the volume of the solid in (c).~~



a) $S = \int_0^1 \sqrt{2}\cos\left(\frac{\pi}{4}x\right)dx - \frac{1}{2}$

$S = \sqrt{2}\left[\frac{\sin\left(\frac{\pi x}{4}\right)}{\pi/4}\right]_0^1 - \frac{1}{2} = \frac{4\sqrt{2}}{\pi}\sin\left(\frac{\pi}{4}\right) - \frac{1}{2} = \frac{4\sqrt{2}}{\pi}\times\frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{4}{\pi} - \frac{1}{2}$ unit

b) $V_T = \int_0^1 \pi y^2\,dx = \pi\int_0^1 2\cos^2\left(\frac{\pi x}{4}\right)dx$

$\cos 2\theta = 2\cos^2\theta - 1$
so $2\cos^2\theta = \cos 2\theta + 1$

$V_T = \pi\int_0^1 \left[\cos\left(\frac{\pi x}{2}\right) + 1\right]dx = \pi\left[\sin\left(\frac{\pi x}{2}\right) + x\right]_0^1 = \pi\left[\sin\frac{\pi}{2} + 1 - (0+0)\right]$

So $V_T = 2\pi$ units^3 But we need to subtract the cone, which volume is $\frac{1}{3}\pi 1^2 \times 1 = \frac{\pi}{3}$ So $V = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ units^3

c) $V = \int_0^{\sqrt{2}} \pi x^2\,dy = \int_0^1 \pi x^2\,dy + \int_1^{\sqrt{2}} \pi x^2\,dy$

$\frac{y}{\sqrt{2}} = \cos\left(\frac{\pi x}{4}\right)$
so $\frac{\pi}{4}x = \cos^{-1}\left(\frac{y}{\sqrt{2}}\right)$
$x = \frac{4}{\pi}\cos^{-1}\left(\frac{y}{\sqrt{2}}\right)$

$V = \int_0^1 \pi y^2\,dy + \int_1^{\sqrt{2}} \pi\frac{16}{\pi^2}\left(\cos^{-1}\left(\frac{y}{\sqrt{2}}\right)\right)^2 dy$

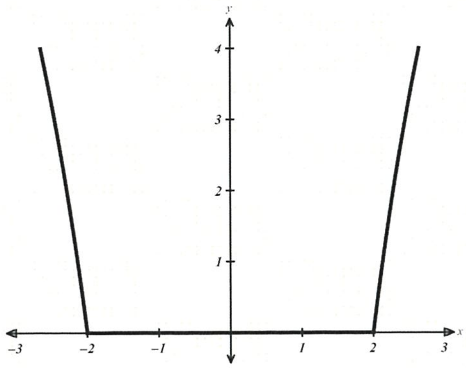
$V = \pi\int_0^1 y^2\,dy + \int_1^{\sqrt{2}} \frac{16}{\pi}\left[\cos^{-1}\left(\frac{y}{\sqrt{2}}\right)\right]^2 dy$

$V = \pi\frac{1}{3} + \frac{16}{\pi}\int_1^{\sqrt{2}}\left[\cos^{-1}\left(\frac{y}{\sqrt{2}}\right)\right]^2 dy$ difficult to calculate.

That's all what you're supposed to do

42 A bowl is formed by rotating the curve $y = 8\,ln(x - 1)$ about the y-axis for $0 \le y \le 4$.

Calculate the capacity of the bowl, giving your answer to one decimal place.



$$V = \int_0^4 \pi\, x^2\, dy$$

$y = 8\ln(x-1)$ so $\ln(x-1) = \frac{y}{8}$, i.e. $x - 1 = e^{y/8} \Leftrightarrow x = 1 + e^{y/8}$

$$V = \pi \int_0^4 \left(1 + e^{y/8}\right)^2 dy$$

$$V = \pi \int_0^4 \left(1 + 2e^{y/8} + \left(e^{y/8}\right)^2\right) dy$$

$$V = \pi \int_0^4 \left[1 + 2e^{y/8} + e^{y/4}\right] dy$$

$$V = \pi \left[y + 2\times 8\, e^{y/8} + 4\, e^{y/4}\right]_0^4$$

$$V = \pi \left[y + 16\, e^{y/8} + 4\, e^{y/4}\right]_0^4$$

$$V = \pi\left[\left(4 + 16e^{16/8} + 4e^{4/4}\right) - \left(0 + 16e^{0/8} + 4e^{0/4}\right)\right]$$

$$V = \pi\left[4 + 16e^2 + 4e - (16 + 4)\right]$$

$$V = \pi\left[16e^2 + 4e - 16\right] = 4\pi\left[4e^2 + e - 4\right] \simeq 66.8 \text{ units}^3$$