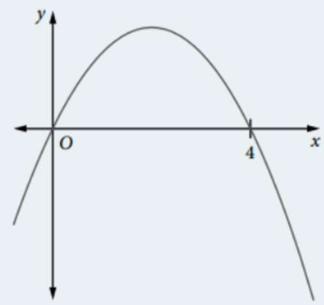
Example 1

A sketch of the function $y = 4x - x^2$ is given in the diagram.



(a) Find $\frac{dy}{dx}$ as a function of x.

(b) Complete the following table of values for $\frac{dy}{dx}$:

x	0	1	2	3	4	5
$\frac{dy}{dx}$						

(c) Draw the graph of $\frac{dy}{dx}$ on the same diagram as a graph of y.

(d) For what values of x is **(i)** $\frac{dy}{dx} > 0$ **(ii)** $\frac{dy}{dx} = 0$ **(iii)** $\frac{dy}{dx} < 0$?

(e) Describe the function $y = 4x - x^2$ where $\frac{dy}{dx} > 0$.

(f) Describe the function $y = 4x - x^2$ where $\frac{dy}{dx} < 0$.

(g) Describe the function $y = 4x - x^2$ where $\frac{dy}{dx} = 0$.

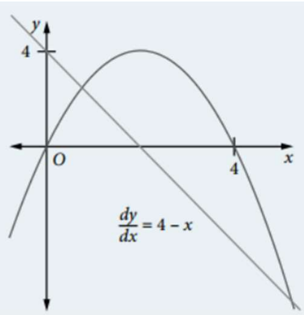
Solution

(a) $\frac{dy}{dx} = 4 - 2x$

(b)

x	0	1	2	3	4	5
$\frac{dy}{dx}$	4	2	0	−2	−4	−6

(c)



(d) **(i)** $x < 2$ **(ii)** $x = 2$ **(iii)** $x > 2$

(e) Where $\frac{dy}{dx} > 0$, y increases as x increases. The curve slopes up.

(f) Where $\frac{dy}{dx} < 0$, y decreases as x increases. The curve slopes down.

(g) Where $\frac{dy}{dx} = 0$, y neither increases nor decreases. It is at its highest point and the tangent at this point is horizontal.

THE SIGN OF THE FIRST DERIVATIVE

The sign of the first derivative

If $\frac{dy}{dx} > 0$ as x increases, the function is an **increasing** function.

If $\frac{dy}{dx} < 0$ as x increases, the function is a **decreasing** function.

If $\frac{dy}{dx} = 0$ at a given value of x, the function is **stationary** at that point; the point is called a **stationary point**. At this point, the tangent to the curve is parallel to the x-axis.

Vice-versa:

If the function is increasing on a given interval, then $\frac{dy}{dx} > 0$ for those values of x.

If the function is decreasing on a given interval, then $\frac{dy}{dx} < 0$ for those values of x.

At a stationary point, $\frac{dy}{dx} = 0$

Example 2

For what values of x is the function $f(x) = 2x^3 - 9x^2 - 24x + 1$:

(a) stationary **(b)** increasing **(c)** decreasing?

Solution

Find $f'(x)$: $f'(x) = 6x^2 - 18x - 24$

Remove common factor: $f'(x) = 6(x^2 - 3x - 4)$

Factorise: $f'(x) = 6(x + 1)(x - 4)$

(a) For stationary points, $f'(x) = 0$: $6(x + 1)(x - 4) = 0$

$x = -1, 4$

Stationary points occur at $x = -1$ or $x = 4$.

(b) Increasing where $f'(x) > 0$: $(x + 1)(x - 4) > 0$

Test $x = 0$: LHS $= 1 \times (-4)$

$= -4$

< 0

Function is not increasing for $-1 < x < 4$

Hence function is increasing for $x < -1$ or $x > 4$.

OR Graphically:

Graph is above axis for $x < -1$ or $x > 4$

Function is increasing for $x < -1$ or $x > 4$

(c) Decreasing where $f'(x) < 0$, so function is decreasing for $-1 < x < 4$.

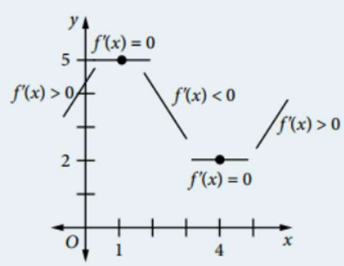
Example 3

Sketch a graph of $f(x)$ such that $f(4) = 2$, $f'(4) = 0$, $f(1) = 5$, $f'(1) = 0$, $f'(x) > 0$ for all $x < 1$ and for all $x > 4$; also $f'(x) < 0$ for $1 < x < 4$.

Solution

Mark this information on a number plane:

Use straight lines to indicate gradients.



Draw a curve using this information:

The only points that you know for sure on $y = f(x)$ are $(1, 5)$ and $(4, 2)$.

From the gradients you know that the curve changes from increasing to decreasing at $(1, 5)$ and from decreasing to increasing at $(4, 2)$. Fit the curve to this information.