MEAN AND VARIANCE OF THE BINOMIAL DISTRIBUTION

Consider a random variable $X \sim B(5, 0.4)$. The probability distribution for X is shown in the following table:

x	0	1	2	3	4	5
$P(X = x)$	0.07776	0.2592	0.3456	0.2304	0.0768	0.01024

Using the method outlined earlier in this chapter, you can find the expected value of X, $E(X)$, and the Variance of X, $\mathrm{Var}(X)$.

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$$\begin{aligned} E(X) &= \sum_{i=0}^{5} x_i p_i \\ &= 0 \times 0.07776 + 1 \times 0.2592 + 2 \times 0.3456 + 3 \times 0.2304 + 4 \times 0.0768 + 5 \times 0.01024 \\ &= 0 + 0.2592 + 0.6912 + 0.6912 + 0.3072 + 0.0512 \\ &= 2 \end{aligned}$$

So, $[E(X)]^2 = 4$

$$\begin{aligned} E(X^2) &= \sum_{i=0}^{5} x_i^2 p_i \\ &= 0 \times 0.07776 + 1 \times 0.2592 + 4 \times 0.3456 + 9 \times 0.2304 + 16 \times 0.0768 + 25 \times 0.01024 \\ &= 0 + 0.2592 + 1.3824 + 2.0736 + 1.2288 + 0.256 \\ &= 5.2 \end{aligned}$$

So, $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$$\begin{aligned} &= 5.2 - 4 \\ &= 1.2 \end{aligned}$$

When dealing with a binomial distribution, if X is the random variable representing the number of successes in n trials and p is the (constant) probability of success, then:

$E(X) = np = \mu$ (mean)

$\mathrm{Var}(X) = np(1 - p)$

$\sigma(X) = \sqrt{np(1-p)}$

Proof: $$E(X) = \sum_{i=0}^{n} i \times {}^{n}C_i\, p^i (1-p)^{n-i} = \sum_{i=1}^{n} i \times {}^{n}C_i\, p^i (1-p)^{n-i}$$

But: $$i\ {}^{n}C_i = i \times \frac{n!}{i!\ (n-i)!} = \frac{n\,(n-1)!}{(i-1)!\ (n-i)!} = \frac{n\,(n-1)!}{(i-1)!\ (n-i)!}$$

So: $$i\ {}^{n}C_i = n\frac{(n-1)!}{(i-1)!\ (n-i+1-1)!} = n\frac{(n-1)!}{(i-1)!\ [(n-1)-(i-1)]!} = n \times {}^{n-1}C_{i-1}$$

Therefore: $$E(X) = \sum_{i=1}^{n} n \times {}^{n-1}C_{i-1}\, p^i (1-p)^{n-i} = n \sum_{i=1}^{n} {}^{n-1}C_{i-1}\, p^i (1-p)^{n-i}$$

We do a change of variable in the summation: $k = i - 1$ So:

$$E(X) = n \sum_{k=0}^{n-1} {}^{n-1}C_k\, p^{k+1} (1-p)^{n-(k+1)} = n \sum_{k=0}^{n-1} {}^{n-1}C_k\, p \times p^k (1-p)^{(n-1)-k}$$

$$E(X) = np \underbrace{\sum_{k=0}^{n-1} {}^{n-1}C_k\, p^k (1-p)^{(n-1)-k}}_{=[p+(1-p)]^{n-1}} = np \times [p + (1-p)]^{n-1} = np \times (1)^{n-1} = np$$

You can check these results for the example given at the very beginning of this section, where $X \sim B(5, 0.4)$. Using the rules:

$$E(X) = np = 5 \times 0.4 = 2$$

$$\mathrm{Var}(X) = np(1-p) = 5 \times 0.4 \times 0.6 = 1.2$$

The same answers are obtained.

Example 8

Find the following statistics for $X \sim B(15, 0.3)$. If necessary, give answers correct to two decimal places.

(a) $E(X)$ **(b)** $\mathrm{Var}(X)$ **(c)** $\sigma(X)$

Solution

(a) $E(X) = np = 15 \times 0.3 = 4.5$

(b) $\mathrm{Var}(X) = np(1-p) = 4.5 \times (1-0.3) = 3.15$

(c) $\sigma(X) = \sqrt{np(1-p)} = \sqrt{3.15} = 1.77$

Example 9

Given $X \sim B(20, p)$ and $E(X) = 5$, find the value of p.

Solution

Use $E(X) = np$: $np = 5$

Substitute the known value: $20 \times p = 5$

Solve for the unknown: $p = \frac{5}{20} = \frac{1}{4} = 0.25$

Example 10

Given $X \sim B(n, p)$, $\mu = 9$ and $\sigma^2 = 6.3$, find the values of n and p.

Solution

$\mu = np$

$\mu = 9 \therefore np = 9$

$\sigma^2 = np(1-p)$

$\sigma^2 = 6.3 \therefore np(1-p) = 6.3$

Substitute for the known value of np in the σ^2 equation: $9(1-p) = 6.3$

$$1 - p = 0.7$$
$$p = 0.3$$

Find n: $n \times 0.3 = 9$

$$n = \frac{9}{0.3}$$
$$n = 30$$

$n = 30$, $p = 0.3$, so $X \sim B(30, 0.3)$