# APPLICATIONS OF DE MOIVRE'S THEOREM

**1** **(a)** Expand $(\cos\theta + i\sin\theta)^4$ by de Moivre's theorem and by the binomial theorem (Pascal's triangle) to show:

**(i)** $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ **(ii)** $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta$

**(b)** Obtain an expression for $\tan 4\theta$ in terms of $\tan\theta$.

**(c)** By making suitable substitutions, solve the following. **(i)** $8x^4 - 8x^2 + 1 = 0$

**(ii)** $16x^4 - 16x^2 + 1 = 0$ **(iii)** $16x^4 - 16x^2 + 3 = 0$ **(iv)** $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

a) $(\cos\theta + i\sin\theta)^4 = (e^{i\theta})^4 = e^{i4\theta} = \cos 4\theta + i\sin 4\theta$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$\_\_\_ = \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$

$\_\_\_ = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

i) So $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ But $\sin^2\theta = 1 - \cos^2\theta$

$\_\_\_ = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$\_\_\_ = \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 + \cos^4\theta - 2\cos^2\theta$

$\_\_\_ = 8\cos^4\theta - 8\cos^2\theta + 1$

ii) $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta$

b) $\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta}{8\cos^4\theta - 8\cos^2\theta + 1}$ divide numerator and denominator by $\cos^4\theta$

$\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{8 - \dfrac{8}{\cos^2\theta} + \dfrac{1}{\cos^4\theta}}$ But $\sin^2\theta + \cos^2\theta = 1$ so $\tan^2\theta + 1 = \dfrac{1}{\cos^2\theta}$

$\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{8 - 8(1+\tan^2\theta) + (1+\tan^2\theta)^2}$

$\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{8 - 8 - 8\tan^2\theta + 1 + 2\tan^2\theta + \tan^4\theta} = \dfrac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1}$

c) i) $x = \cos\theta$ $8x^4 - 8x^2 + 1 = 0$ $\Leftrightarrow \cos 4\theta = 0$

so $4\theta = \pm\dfrac{\pi}{2} + 2\pi n$ $\theta = \pm\dfrac{\pi}{8} + \dfrac{n\pi}{2}$ $x = \cos\left(\pm\dfrac{\pi}{8} + \dfrac{n\pi}{2}\right)$

$x = \cos\dfrac{\pi}{8}$ or $x = \cos\dfrac{5\pi}{8} = -\cos\dfrac{3\pi}{8}$ or $x = \cos\dfrac{3\pi}{8}$ or $x = \cos\left(-\dfrac{9\pi}{8}\right) = -\cos\dfrac{\pi}{8}$

So $x = \pm\cos\left(\dfrac{\pi}{8}\right)$ or $x = \pm\cos\left(\dfrac{3\pi}{8}\right)$

c) (ii) $16x^4 - 16x^2 + 1 = 0 \iff 2[8x^4 - 8x^2 + 1] - 1 = 0$

$\iff 8x^4 - 8x^2 + 1 = 1/2$

let $x = \cos\theta$ so the equation is equivalent to $\cos 4\theta = 1/2 = \cos\frac{\pi}{3}$

So $4\theta = \pm\frac{\pi}{3} + 2n\pi$ $\quad \theta = \pm\frac{\pi}{12} + n\frac{\pi}{2}$

Solutions are $x = \cos\frac{\pi}{12}$ or $x = \cos\left(\frac{7\pi}{12}\right) = -\cos\left(\frac{5\pi}{12}\right)$

or $x = \cos\left(\frac{5\pi}{12}\right)$ or $x = \cos\left(-\frac{13\pi}{12}\right) = -\cos\left(\frac{\pi}{12}\right)$

So $x = \pm\cos\frac{\pi}{12}$ and $x = \pm\cos\left(\frac{5\pi}{12}\right)$ are solutions.

iii) $16x^4 - 16x^2 + 3 = 0 \iff 2[8x^4 - 8x^2 + 1] + 1 = 0$

$\iff 8x^4 - 8x^2 + 1 = -1/2$

Let $x = \cos\theta$ so the equation is equivalent to $\cos 4\theta = -1/2 = \cos\frac{2\pi}{3}$

So $4\theta = \pm\frac{2\pi}{3} + 2n\pi$ $\quad \theta = \pm\frac{2\pi}{12} + n\frac{\pi}{2} = \pm\frac{\pi}{6} + n\frac{\pi}{2}$

Solutions are $x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ or $x = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

or $x = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ or $x = \cos\left(\frac{7\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

So $x = \pm\frac{1}{2}$ and $x = \pm\frac{\sqrt{3}}{2}$ are solutions.

iv) $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0 \iff x^4 - 6x^2 + 1 = 4x - 4x^3$

$\iff \dfrac{4x - 4x^3}{x^4 - 6x^2 + 1} = 1 \iff \tan 4\theta = 1$ with $\tan\theta = x$ (see part b)

$\tan 4\theta = 1 = \tan \pi/4$

So $4\theta = \frac{\pi}{4} + n\pi$ $\quad \theta = \frac{\pi}{16} + n\frac{\pi}{4}$

$x = \tan\left(\frac{\pi}{16}\right)$ or $x = \tan\left(\frac{5\pi}{16}\right)$ or $x = \tan\left(\frac{9\pi}{16}\right)$ or $x = \tan\left(\frac{13\pi}{16}\right)$

i.e. $x = -\tan\left(\frac{7\pi}{16}\right)$ $\quad x = -\tan\left(\frac{3\pi}{16}\right)$

**2** **(a)** Given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ (see Example **19**, page 20), solve $8x^3 - 6x - 1 = 0$.

**(b)** Show that $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$.

a/ $8x^3 - 6x - 1 = 0 \iff 4x^3 - 3x - \frac{1}{2} = 0$

let $x = \cos\theta$ $\iff \underbrace{4\cos^3\theta - 3\cos\theta}_{=\cos 3\theta} - \frac{1}{2} = 0$

$\iff \cos 3\theta = 1/2 = \cos\left(\frac{\pi}{3}\right)$

So $3\theta = \pm\frac{\pi}{3} + 2n\pi$ so $\theta = \pm\frac{\pi}{9} + \frac{2n\pi}{3}$

So $x = \cos\frac{\pi}{9}$ or $x = \cos\left(\frac{7\pi}{9}\right) = -\cos\left(\frac{2\pi}{9}\right)$

or $x = \cos\left(\frac{13\pi}{9}\right) = -\cos\left(\frac{4\pi}{9}\right)$

b) We know the product of the 3 roots $\alpha, \beta, \gamma$ is equal to $-\frac{d}{a}$, i.e $\alpha\beta\gamma = -\frac{d}{a} = -\frac{(-1/2)}{4} = \frac{1}{8}$

So $\cos\left(\frac{\pi}{9}\right) \times \left(-\cos\left(\frac{2\pi}{9}\right)\right) \times \left(-\cos\left(\frac{4\pi}{9}\right)\right) = \frac{1}{8}$

or $\cos\left(\frac{\pi}{9}\right) \times \cos\left(\frac{2\pi}{9}\right) \times \cos\left(\frac{4\pi}{9}\right) = \frac{1}{8}$

# APPLICATIONS OF DE MOIVRE'S THEOREM

**3** Let $z = \cos\theta + i\sin\theta$.

**(a)** Show that: **(i)** $z^n + z^{-n} = 2\cos n\theta$ **(ii)** $z^n - z^{-n} = 2i\sin n\theta$

**(b)** Show that $(z - z^{-1})^3 = (z^3 - z^{-3}) - 3(z - z^{-1})$ **(c)** Hence show that $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.

a) $z^n = e^{in\theta}$ (De Moivre formula) and $e^{in\theta} = \cos n\theta + i\sin n\theta$

$z^{-n} = e^{-in\theta} = \left(e^{-i\theta}\right)^n$ and $e^{-in\theta} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta)$

i) Adding the two equations, we obtain:

$$2\cos n\theta = e^{in\theta} + e^{-in\theta} = z^n + z^{-n}$$

$$\text{So } \cos n\theta = \frac{z^n + z^{-n}}{2}$$

ii) Subtracting both equations, we obtain $z^n - z^{-n} = 2i\sin n\theta$.

b) $(z - z^{-1})^3 = z^3 - 3z^2z^{-1} + 3zz^{-2} - z^{-3}$ (expanding).

$= (z^3 - z^{-3}) - 3(z - z^{-1})$

c) $(z - z^{-1})^3 = (2i\sin\theta)^3 = -8i\sin^3\theta$.

whereas $(z^3 - z^{-3}) - 3(z - z^{-1}) = (2i\sin 3\theta) - 3(2i\sin\theta)$

$= 2i\sin 3\theta - 6i\sin\theta$

So $-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$-4\sin^3\theta = \sin 3\theta - 3\sin\theta$

$$\sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$$

## APPLICATIONS OF DE MOIVRE'S THEOREM

4 (a) Let $z = \cos\theta + i\sin\theta$ and let $w = z + \frac{1}{z}$. Given $z^n + z^{-n} = 2\cos n\theta$, prove that
$w^3 - 2w^2 - w + 2 = \left(z^3 + \frac{1}{z^3}\right) - 2\left(z^2 + \frac{1}{z^2}\right) + 2\left(z + \frac{1}{z}\right) - 2.$

(b) Hence solve $\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1 = 0$ for $-\pi \le \theta \le \pi$.

a) $w^3 - 2w^2 - w + 2 = \left(z + \frac{1}{z}\right)^3 - 2\left(z + \frac{1}{z}\right)^2 - \left(z + \frac{1}{z}\right) + 2$

$= z^3 + 3z^2\frac{1}{z} + 3z\frac{1}{z^2} + \frac{1}{z^3} - 2\left[z^2 + 2 + \frac{1}{z^2}\right] - z - \frac{1}{z} + 2$

$= z^3 + 3z + 3 \times \frac{1}{z} + \frac{1}{z^3} - 2z^2 - 4 - 2 \times \frac{1}{z^2} - z - \frac{1}{z} + 2$

$= \left[z^3 + \frac{1}{z^3}\right] - 2\left[z^2 + \frac{1}{z^2}\right] + 2z + 2 \times \frac{1}{z} - 2$

$= \left[z^3 + \frac{1}{z^3}\right] - 2\left[z^2 + \frac{1}{z^2}\right] + 2\left[z + \frac{1}{z}\right] - 2$

b) $\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1 = 0$

$\left[z^3 + \frac{1}{z^3}\right] - 2\left[z^2 + \frac{1}{z^2}\right] + 2\left[z + \frac{1}{z}\right] - 2 = 2\cos 3\theta - 2 \times 2\cos 2\theta + 2 \times 2\cos\theta - 2$

$= 2\left[\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1\right]$

So solving $\left[\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1\right] = 0$ is the same than solving $w^3 - 2w^2 - w + 2 = 0$ with $w = z + \frac{1}{z} = 2\cos\theta$
equation ①

Equation ① has one obvious solution $w = 1$

① $\Leftrightarrow (w-1)(w^2 - w - 2) = 0$

$\Delta = 1 - 4 \times (-2) \times 1 = 9 = 3^2$

$w_1 = \frac{1+3}{2} = 2$ or $w_2 = \frac{1-3}{2} = -1$

So either: 1) $w = 1 = z + \frac{1}{z} = 2\cos\theta \rightsquigarrow \cos\theta = \frac{1}{2}$ so $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$

2) $w = 2 = z + \frac{1}{z} = 2\cos\theta$ so $\cos\theta = 1$ so $\theta = 0$

3) $w = -1$ so $2\cos\theta = -1 \rightsquigarrow \cos\theta = -\frac{1}{2}$ so $\theta = \frac{2\pi}{3}$ or $\theta = \frac{-2\pi}{3}$

**5** Express $\cos 3\theta$ and $\cos 2\theta$ in terms of $\cos\theta$. Show that the equation $\cos 3\theta = \cos 2\theta$ can be expressed as $4x^3 - 2x^2 - 3x + 1 = 0$, where $x = \cos\theta$. By solving this equation for $x$, find the exact value of $\cos\frac{2\pi}{5}$.

$$e^{i3\theta} = (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 = \cos^3\theta + \cos\theta \times 3 \times (i\sin\theta)^2 + i(\ldots)$$

$$= \cos 3\theta + i\sin 3\theta$$

$$= \cos^3\theta - 3\cos\theta(\sin^2\theta) + i(\ldots)$$

$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) + i(\ldots)$$

So $\cos 3\theta = \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$\qquad = 4\cos^3\theta - 3\cos\theta$

and $\cos 2\theta = 2\cos^2\theta - 1$

$\cos 3\theta = \cos 2\theta \iff 4\cos^3\theta - 3\cos\theta = 2\cos^2\theta - 1$

Say $x = \cos\theta$ $\quad \iff 4x^3 - 2x^2 - 3x + 1 = 0$

$x = 1$ is an obvious root so $(x-1)(4x^2+2x-1) = 0$

so let's solve the quadratic: $\Delta = 2^2 - 4 \times (-1) \times 4 = 4 + 16 = 20$

$\sqrt{\Delta} = 2\sqrt{5}$

$x = \dfrac{-2 \pm 2\sqrt{5}}{8} = \dfrac{-1 \pm \sqrt{5}}{4}$ are the roots.

So either $\cos\theta = 1$ i.e. $\theta = 0 + 2n\pi$

or $x = \cos\theta = \dfrac{-1 \pm \sqrt{5}}{4}$

From $\cos 3\theta = \cos 2\theta$

we obtain $3\theta = \pm 2\theta + 2n\pi$

So $5\theta = 2n\pi$ or $\theta = 2n\pi$

$\theta = \dfrac{2n\pi}{5}$

So $\cos\left(\dfrac{2\pi}{5}\right) = \dfrac{-1+\sqrt{5}}{4} > 0$ in the 1st quadrant.

# APPLICATIONS OF DE MOIVRE'S THEOREM

**6** **(a)** Use de Moivre's theorem to express $\cos 4\theta$ in terms of $\cos\theta$.

**(b)** Use your result from part **(a)** to solve the equation $8x^4 - 8x^2 + 1 = 0$.

**(c)** Show that: $\cos\frac{\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{7\pi}{8} = 0$.

**(d)** Show that: $\cos\frac{\pi}{8}\cos\frac{3\pi}{8}\cos\frac{5\pi}{8}\cos\frac{7\pi}{8} = \frac{1}{8}$.

a) $(e^{i\theta})^4 = e^{i4\theta} = \cos 4\theta + i \sin 4\theta$.

so $\cos 4\theta$ is equal to the real part of $(\cos\theta + i\sin\theta)^4$

which is $\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$\cos 4\theta = \cos^4\theta + 6\cos^4\theta - 6\cos^2\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$\underline{\quad} = 8\cos^4\theta - 8\cos^2\theta + 1$$

b) $8x^4 - 8x^2 + 1 = 0 \iff \cos 4\theta = 0 = \cos\left(2n\pi \pm \frac{\pi}{2}\right)$

$x = \cos\theta$ So $4\theta = \pm\frac{\pi}{2} + 2n\pi$

$$\theta = \pm\frac{\pi}{8} + n\frac{\pi}{2}$$

Solutions are $\cos\left(\frac{\pi}{8}\right)$, $\cos\left(\frac{3\pi}{8}\right)$, $\cos\left(\frac{5\pi}{8}\right)$ and $\cos\left(\frac{7\pi}{8}\right)$

The sum of roots of a quartic equation $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$

So $\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{3\pi}{8}\right) + \cos\left(\frac{5\pi}{8}\right) + \cos\left(\frac{7\pi}{8}\right) = 0$ as $b = 0$

The product of roots of a quartic equation is $\alpha\beta\gamma\delta = \frac{e}{a} = \frac{1}{8}$

So $\cos\left(\frac{\pi}{8}\right) \times \cos\left(\frac{3\pi}{8}\right) \times \cos\left(\frac{5\pi}{8}\right) \times \cos\left(\frac{7\pi}{8}\right) = \frac{1}{8}$

# APPLICATIONS OF DE MOIVRE'S THEOREM

7 **(a)** Use de Moivre's theorem to express $\cos 5\theta$ and $\sin 5\theta$ as powers of $\cos\theta$ and $\sin\theta$.

**(b)** Hence express $\tan 5\theta$ as a rational function of $t$ where $t = \tan\theta$.

**(c)** By considering the roots of $\tan 5\theta = 0$, deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$.

a) $\cos 5\theta$ is the real part of $e^{i5\theta}$ which is $(e^{i\theta})^5$

so we need to find the real part of $(e^{i\theta})^5$

1 4 6 4 1

1 5 10 10 5 1

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$

$$= \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

$$= [\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta] + i[5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta]$$

So $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

and $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

b) $$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

we divide numerator and denominator by $\cos^5\theta$

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

c) roots of $\tan 5\theta = 0$ are $5\theta = n\pi$ or $\theta = \frac{n\pi}{5}$

So $\tan 5\theta = 0$ is equivalent to $5t - 10t^3 + t^5 = 0$

or $(5 - 10t^2 + t^4) = 0$

$\alpha\beta\gamma\delta = \frac{e}{a} = \frac{5}{1} = 5$ (product of roots for quartic equation)

so $\tan\left(\frac{\pi}{5}\right) \times \tan\left(\frac{2\pi}{5}\right) \times \tan\left(\frac{3\pi}{5}\right) \times \tan\left(\frac{4\pi}{5}\right) = 5$