EXPONENTIAL GROWTH AND DECAY

1 If $\frac{dy}{dx} = 2y$ and $y = 5$ where $x = 0$, express y as a function of x.

The general solution is $y = Ae^{2x}$

when $x = 0$, $y = 5$, so $5 = Ae^{0} = A$ so $A = 5$

$\therefore\ y = 5e^{2x}$ [Note: indeed $\frac{dy}{dx} = 10e^{2x} = 2 \times y$]

2 If $\frac{dN}{dt} = -0.5N$ and $N = 100$ when $t = 0$, then N expressed as a function of t is:

A $N = 100e^{0.5t}$ **B** $N = 100e^{-0.5t}$ **C** $N = 0.5e^{100t}$ **D** $N = 0.5e^{-100t}$

The general solution is $N(t) = Ae^{-0.5t}$

when $t = 0$, $N = 100$, so $100 = Ae^{-0.5 \times 0} = A$

so $N(t) = 100e^{-0.5t}$ $\boxed{B}$

[Note: indeed $\frac{dN}{dt} = 100 \times (-0.5)e^{-0.5t} = -0.5N(t)$]

4 If $\frac{dy}{dt} = -3y$ and $y = 20$ when $t = 0$, express y as a function of t.

The general solution is $y = Ae^{-3t}$

when $t = 0$, $y = 20$, so $20 = Ae^{-0} = A$ so $A = 20$

$y = 20e^{-3t}$

[Note: indeed $\frac{dy}{dt} = 20 \times (-3)e^{-3t} = -3 \times y$]

7 If $N = Ae^{kt}$, $N = 200$ when $t = 0$, and $N = 1478$ when $t = 5$, find the values of A and k.

At $t = 0$ $N = 200$ so $200 = Ae^{0} = A$ so $A = 200$.

and $N = 200e^{kt}$

At $t = 5$ $N = 1478$ so $1478 = 200e^{5k}$ so $e^{5k} = \frac{1478}{200} = 7.39$

so $\ln e^{5k} = \ln 7.39$ so $5k = \ln 7.39$

so $k = \frac{\ln 7.39}{5}$

9 The population of a city increases at a rate that is proportional to the current population. If the population of the city was 100 000 in the year 2000 and 120 000 in the year 2010, express the population P in terms of t years after 2000.

$\frac{dP}{dt} = kP$ so $P = Ae^{kt}$

At $t=0$ (i.e. in 2000) $P = 100{,}000 = Ae^0$ so $A = 100{,}000$

$P = 100{,}000\, e^{kt}$

At $t = 10$ (i.e. in 2010) $P = 100{,}000\, e^{10k} = 120{,}000$

so $e^{10k} = \frac{120{,}000}{100{,}000} = \frac{12}{10} = 1.2$ so $10k = \ln 1.2$

so $k = \frac{\ln 1.2}{10}$ $P(t) = 100{,}000\, e^{\left(\frac{\ln(1.2)}{10}\right)t} = 100{,}000\, e^{\ln 1.2 \times \frac{t}{10}}$

$P(t) = 100{,}000 \left(e^{\ln 1.2}\right)^{t/10} = 100{,}000 \times 1.2^{t/10}$

11 The rate of decay of a radioactive isotope is proportional to the amount of the isotope present at any time. If one-half of a given quantity of the isotope decays in 1600 years, what percentage will decay in 100 years?

$\frac{dQ}{dt} = kQ$ so $Q = Q_0 e^{kt}$ is the general solution.

where Q_0 is the quantity present at $t=0$

At $t = 1{,}600$ $Q = \frac{Q_0}{2}$ so $\frac{Q_0}{2} = Q_0 e^{1600k}$

so $e^{1600k} = \frac{1}{2}$ $\Leftrightarrow$ $1{,}600k = \ln\left(\frac{1}{2}\right) = -\ln 2$

so $k = -\frac{\ln 2}{1{,}600}$ or $Q = Q_0 e^{-\frac{\ln 2}{1600}t}$

$Q = Q_0 e^{\ln 2\left(\frac{-t}{1600}\right)} = Q_0 \left[e^{\ln 2}\right]^{-t/1600} = Q_0 \times 2^{-t/1600}$

for $t = 100$ $Q = Q_0\, 2^{-100/1600}$ so $\frac{Q}{Q_0} = 2^{-1/16} = \frac{1}{2^{1/16}}$

So $\frac{Q}{Q_0} \simeq 0.957$ so about 0.042 (or 4.2%) will have decayed in 100 years

12 The number of bacteria N in a colony after t minutes is given by $N = 10000e^{0.05t}$. Find:

(a) the number of bacteria after 10 minutes
(b) the time required for the original number to double.
(c) Find the rate at which the colony increases when: (i) $t = 10$ (ii) $N = 20000$

a) At $t = 10$ $N(10) = 10{,}000\, e^{0.05\times 10} = 10{,}000\, e^{0.5} = 10000\sqrt{e} = 16{,}487$

b) For $N = 20{,}000$, we must have $e^{0.05t} = 2$

so $t = \dfrac{\ln 2}{0.05} = 13.8629$ so $13'52''$ approx

c) $N(t) = 10{,}000\, e^{0.05t}$ so $\dfrac{dN}{dt} = 10{,}000 \times 0.05\, e^{0.05t} = 500\, e^{0.05t}$

i) At $t = 10$ $\dfrac{dN}{dt} = 500\, e^{0.05\times 10} = 500\, e^{0.5} = 500\sqrt{e} = 824$ bacteria/min

ii) when $N = 20{,}000$: $\dfrac{dN}{dt} = 10{,}000 \times 0.05\, e^{0.05t} = 0.05\, N$

so $\dfrac{dN}{dt} = 0.05 \times 20{,}000 = 1000$ bacteria /min.

13 A vessel filled with liquid is being emptied. The volume V cubic metres of liquid remaining after t minutes is given by $V = V_0 e^{-kt}$.

(a) Show that $\dfrac{dV}{dt} = -kV$.
(b) If one-quarter of the vessel is emptied in the first 5 minutes, what fraction remains after 10 minutes?
(c) At what rate is the liquid flowing out:
 (i) after 10 minutes (ii) when one-quarter of the vessel is empty.

a) $V = V_0\, e^{-kt}$ so $\dfrac{dV}{dt} = -k V_0\, e^{-kt} = -kV$

b) At $t = 5$, $V = \dfrac{3V_0}{4}$ so $\dfrac{3V_0}{4} = V_0\, e^{-5k}$ so $e^{-5k} = 3/4$

so $-5k = \ln 3/4 = -\ln 4/3$ so $k = \dfrac{\ln(4/3)}{5}$

so $V = V_0 \left(\dfrac{4}{3}\right)^{-t/5}$ so at $t = 10$, $\dfrac{V}{V_0} = \left(\dfrac{4}{3}\right)^{-2} = \left(\dfrac{3}{4}\right)^2 = \dfrac{9}{16}$

c) i) $\dfrac{dV}{dt} = -kV = -kV_0 \left(\dfrac{4}{3}\right)^{t/5}$ so at $t = 10$ $\dfrac{dV}{dt} = -\dfrac{\ln(4/3)}{5} V_0 \left(\dfrac{4}{3}\right)^{-2}$

$\dfrac{dV}{dt} = -\dfrac{\ln(4/3)}{5}\left(\dfrac{3}{4}\right)^2 V_0 = -0.032\, V_0$

ii) When $V = \dfrac{3V_0}{4}$ (i.e $t = 5$ as stated at b)) $\dfrac{dV}{dt} = -\dfrac{\ln(4/3)}{5} V_0 \left(\dfrac{4}{3}\right)^{-1}$

so $\dfrac{dV}{dt} = -\dfrac{\ln(4/3)}{5}\left(\dfrac{3}{4}\right) V_0 \simeq -0.043\, V_0$

14 For a period of its life, the increase in the diameter of a tree approximately follows the rule $D(t) = Ae^{kt}$, where $D(t)$ is the diameter of the tree t years after the beginning of this period.

(a) If the diameter is initially 50 cm, find the value of A.
(b) If $D'(t) = 0.1D(t)$, find the value of k.
(c) After how many years is the diameter 61 cm?

a) At $t=0$ $\quad D(0) = 50 = Ae^0 = A$ so $A = 50$.

b) $D(t) = 50e^{kt}$ so $D'(t) = k50e^{kt} = k\,D(t)$. so $k = 0.1$

c) $D(t) = 50\,e^{0.1t}$

For $D = 61$, we must have $61 = 50\,e^{0.1t}$

so $e^{0.1t} = \frac{61}{50}$ so $0.1t = \ln\frac{61}{50}$ so $t = 10\ln(61/50)$

$t = 1.988$ so 2 years approx.

15 The charge Q (measured in coulombs) on the plate of a condenser t seconds after it starts to discharge is given by the formula $Q = Ae^{-kt}$.

(a) If the original charge is 5000 coulombs, find the value of A.
(b) If $\frac{dQ}{dt} = -2000$ when $Q = 1000$, find the value of k.
(c) Find the rate of discharge when $Q = 5000$.

a) At $t=0$ $\quad Q(0) = Ae^0 = A = 5000$ so $A = 5{,}000$.

b) $\frac{dQ}{dt} = A \times (-k)\,e^{-kt} = -kQ$ so $k = -\frac{dQ/dt}{Q} = -\frac{-2000}{1000} = 2$

so $\boxed{k = 2}$

c) so $Q = 5000\,e^{-2t}$

$\frac{dQ}{dt} = -k \times 5{,}000\,e^{-2t} = -kQ = -2Q.$

so when $Q = 5000$, $\frac{dQ}{dt} = -10{,}000$

the rate of discharge is $10{,}000\ C\,s^{-1}$

EXPONENTIAL GROWTH AND DECAY

16 The rate of increase in the number N of bacteria in a certain culture is given by $\frac{dN}{dt} = 0.15N$, where t is time in hours.

(a) If the original number of bacteria is 1000, express N as a function of t.
(b) After how many hours has the original number of bacteria doubled? What is the rate of increase at this time?

a) At $t=0$, $N = 1000$ $\qquad N(t) = A\,e^{0.15t}$

so $1{,}000 = A\,e^{0} = A$ so $N(t) = 1{,}000\,e^{0.15t}$

b) For $N = 2000$, we must have $2000 = 1000\,e^{0.15t}$

so $e^{0.15t} = 2$ so $0.15t = \ln 2$ or $t = \frac{\ln 2}{0.15}$

$t = 4.62$ hours or 4 hours 37′ $\qquad$ At that time:

$\frac{dN}{dt} = 0.15N = 0.15 \times 2000 = 300$ bacteria / hour

17 Sunlight transmitted into water loses intensity as it penetrates to greater depths according to the law $I(d) = I(0)e^{-kd}$, where $I(d)$ is the intensity at depth d metres below the surface. If $I(300) = 0.3I(0)$, find:

(a) the value of k **(b)** the depth at which the intensity would be decreased by one-half.

a) $I(300) = I(0)\,e^{-300k} = 0.3\ I(0)$

so $e^{-300k} = 0.3$ so $-300k = \ln 0.3$ so $k = -\frac{\ln 0.3}{300}$

$k = -\frac{\ln(3/10)}{300} = \frac{\ln(10/3)}{300} \simeq 0.004$

b) For $I(d) = \frac{I(0)}{2}$, we must have $\frac{I(0)}{2} = I(0)\,e^{-0.004d}$

so $e^{-0.004d} = \frac{1}{2}$ so $-0.004d = \ln\frac{1}{2} = -\ln 2$

so $d = \frac{\ln 2}{0.004} = 173$ m approx

18 The rate of increase of the population $P(t)$ of a particular island is given by the equation $\frac{d}{dt}P(t) = kP(t)$, where t is time in years. In the year 2000 the population was 1000 and in 2010 it had decreased to 800.

(a) Find k, the annual growth rate. **(b)** In how many years will the population be half that in 2000?

a) $P(t) = Ae^{kt}$ so $A = \frac{P(t)}{e^{kt}} = \frac{1000}{e^{k2000}} = \frac{800}{e^{k \times 2010}}$

so $e^{k(2010-2000)} = \frac{800}{1000} = \frac{4}{5}$ so $e^{10k} = 4/5$ so $k = \frac{\ln(4/5)}{10} \simeq -0.02$

b) So $P = Ae^{kt} = \frac{1000}{e^{\frac{\ln(4/5) \times 2000}{10}}} e^{\left(\frac{\ln 4/5}{10}\right)t} = \frac{1000}{(4/5)^{200}} \left(\frac{4}{5}\right)^{t/10}$

For $P = 500$, we must have $500 = \frac{1000}{(4/5)^{200}} \left(\frac{4}{5}\right)^{t/10}$

so $\left(\frac{4}{5}\right)^{t/10} = \frac{1}{2}\left(\frac{4}{5}\right)^{200}$

so $\frac{t}{10} \ln\left(\frac{4}{5}\right) = \ln\left[\frac{1}{2}\left(\frac{4}{5}\right)^{200}\right] = 200 \ln\left(\frac{4}{5}\right) - \ln 2$

so $t = 10\left[200 - \frac{\ln 2}{\ln(4/5)}\right] = 2031$

19 A substance decomposes at a rate equal to k times the mass of the substance present. If initially the mass is M, find the mass m at time t. If $k = 0.1$, find the value of t for which $m = \frac{M}{2}$.

$\frac{dm}{dt} = -km$ so $m(t) = A \cdot e^{-kt} = Ae^{-0.1t}$

At $t = 0$ $m = M$ so $A = M$ $m(t) = Me^{-0.1t}$

For $m = \frac{M}{2}$, we must have $\frac{M}{2} = Me^{-0.1t}$

so $e^{-0.1t} = \frac{1}{2}$ so $\ln e^{-0.1t} = \ln\frac{1}{2} = -\ln 2$

so $-0.1t = -\ln 2$ so $t = \frac{\ln 2}{0.1} = 10 \ln 2 = 6.93$ units

20 A heated body is cooling. The excess of its temperature above that of its surroundings is $\theta = Ae^{-kt}$, where θ is measured in °C and t is in minutes.

(a) At time $t = 0$, $\theta = 80$. Find A.

(b) If the temperature of the surroundings is 20°C and the body cools to 70°C in 10 minutes, find:

(i) the body's temperature after 20 minutes   (ii) the time taken to reach 60°C.

a) $80 = A\,e^0$ so $A = 80$ $\theta = 80\,e^{-kt}$

b) $\theta = 50$ for $t = 10$. so $50 = 80\,e^{-10k}$

so $e^{-10k} = 5/8$ so $k = \frac{1}{10}\ln(8/5) \simeq 0.047$

i) At $t = 20$, $\theta = 80\,e^{-0.047 \times 20} = 31.25$ so the temperature of the body is $31.25 + 20 = 51.25°C$

ii) For $\theta = 40°$, we must have $40 = 80\,e^{-0.047t}$

so $e^{-0.047t} = 0.5$ so $t = -\frac{\ln 0.5}{0.047} = \frac{\ln 2}{0.047} \simeq 14.75$

so $14'45''$

21 The number N of bacteria in a colony grows according to the rule $\frac{dN}{dt} = kN$. If the original number increases from 4000 to 8000 in 4 days, find the number after another 4 days.

so $N = Ae^{kt}$

At $t = 0$, $N = 4000$ so $N(t) = 4000\,e^{kt}$

At $t = 4$ $N = 8{,}000$ so $8{,}000 = 4{,}000\,e^{4k}$

so $e^{4k} = 2 \iff k = \frac{\ln 2}{4}$ so $N(t) = 4{,}000\,e^{\ln 2 \times \frac{t}{4}}$

$N(t) = 4{,}000 \times 2^{t/4}$

At $t = 8$, $N(8) = 4{,}000 \times 2^{8/4} = 4{,}000 \times 2^2 = 16{,}000$ bacteria.

22 A population of size N is decreasing according to the rule $\frac{dN}{dt} = -\frac{N}{100}$, where t is the time in days. If the population is initially of size N_0, find how much time it takes for the size to be halved, to the next day.

$$N(t) = N_0 e^{-t/100}$$

So $N = \frac{N_0}{2}$ when $\frac{N_0}{2} = N_0 e^{-t/100} \Rightarrow e^{-t/100} = \frac{1}{2}$

$\Leftrightarrow -\frac{t}{100} = \ln \frac{1}{2} \Leftrightarrow t = 100 \ln 2$

$t = 69.3$ days.

So 70 days, to the next day

23 A radioactive substance decays at a rate that is proportional to the mass of radioactive substance present at any time. If 10% decays in 200 years, what percentage of the original radioactive mass will remain after 1000 years?

$\frac{dN}{dt} = -kN$ so $N(t) = N_0 e^{-kt}$

At $t = 200$ $\frac{N}{N_0} = 0.9$ so $0.9 = e^{-200k}$

so $\ln 0.9 = -200k$ so $k = -\frac{\ln 9/10}{200} = \frac{\ln(10/9)}{200}$

so $N(t) = N_0 e^{\frac{\ln(9/10)}{200} t} = N_0 \left(\frac{9}{10}\right)^{t/200}$

At $t = 1000$, $\frac{N}{N_0} = \left(\frac{9}{10}\right)^{\frac{1000}{200}} = \left(\frac{9}{10}\right)^5 \simeq 0.59$

so 59% of the original radioactive mass remains.