TRIGONOMETRIC EQUATIONS INVOLVING ANGLE FORMULAE

1 Solve: **(a)** $\cos 2\theta = \cos\theta,\ 0 \le \theta \le 2\pi$ **(b)** $2\cos 2\theta = 4\cos\theta - 3,\ 0 \le \theta \le 2\pi$

a) $\Leftrightarrow \cos 2\theta - \cos\theta = 0$ $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

if $\begin{cases} A-B = \alpha \\ A+B = \beta \end{cases}$ then $2A = \alpha+\beta \Rightarrow A = \dfrac{\alpha+\beta}{2}$

and $2B = \beta - \alpha \Rightarrow B = \dfrac{\beta-\alpha}{2}$

$\cos\alpha - \cos\beta = 2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\beta-\alpha}{2}\right)$

so $\cos 2\theta - \cos\theta = 0 \Leftrightarrow 2\sin\dfrac{3\theta}{2}\sin\left(\dfrac{\theta-2\theta}{2}\right) = 0$

$\Leftrightarrow \sin\dfrac{3\theta}{2}\sin\left(-\dfrac{\theta}{2}\right) = 0 \Leftrightarrow \sin\left(\dfrac{3\theta}{2}\right)\sin\left(\dfrac{\theta}{2}\right) = 0$

so either $\sin\dfrac{3\theta}{2} = 0 = \sin 0 \therefore \dfrac{3\theta}{2} = 0\times(-1)^n + n\pi \Leftrightarrow \theta = \dfrac{2n\pi}{3}$

$n=0$ gives $\theta = 0$; $n=1$ gives $\theta = \dfrac{2\pi}{3}$; $n=2$ gives $\theta = \dfrac{4\pi}{3}$

$n=3$ gives $\theta = 2\pi$

OR $\sin\left(\dfrac{\theta}{2}\right) = 0 \quad \therefore \dfrac{\theta}{2} = n\pi \Leftrightarrow \theta = 2n\pi$

4 solutions $0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, 2\pi$

b) $\Leftrightarrow 2(2\cos^2\theta - 1) - 4\cos\theta + 3 = 0$

$\Leftrightarrow 4\cos^2\theta - 4\cos\theta + 1 = 0 \Leftrightarrow 4X^2 - 4X + 1 = 0$

$\Delta = 16 - 4\times 4 = 0$ So 1 solution $X = \dfrac{-(-4)}{2\times 4} = \dfrac{1}{2}$

$\cos\theta = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

General solution is $\theta = \pm\dfrac{\pi}{3} + 2n\pi$

$n=0$ gives $\theta = \dfrac{\pi}{3}$

$n=1$ gives $\theta = \dfrac{\pi}{3} + 2\pi = \dfrac{7\pi}{3}$ (outside of interval $[0, 2\pi]$)

and $\theta = -\dfrac{\pi}{3} + 2\pi = \dfrac{5\pi}{3}$

Two solutions: $\dfrac{\pi}{3}$ and $\dfrac{5\pi}{3}$

TRIGONOMETRIC EQUATIONS INVOLVING ANGLE FORMULAE

1 Solve:

(c) $3\tan 2\theta = 2\tan\theta,\ 0 \le \theta \le 2\pi$

(d) $\tan\theta + 2\cot\theta = 3,\ 0° \le \theta \le 360°$

c) $\Leftrightarrow 3 \times \dfrac{2\tan\theta}{1-\tan^2\theta} = 2\tan\theta \Leftrightarrow 6\tan\theta = 2\tan\theta - 2\tan^3\theta$

$\Leftrightarrow 4\tan\theta + 2\tan^3\theta = 0 \Leftrightarrow \tan\theta(2+\tan^2\theta) = 0$

either $\tan\theta = 0$ $\quad \theta = 0 \quad \theta = \pi$ or $\theta = 2\pi$

OR $2 + \tan^2\theta = 0$ which is impossible as $\tan^2\theta > 0$

So 3 solutions $0, \pi, 2\pi$

d) $\Leftrightarrow \tan\theta + \dfrac{2}{\tan\theta} = 3 \Leftrightarrow \tan^2\theta + 2 = 3\tan\theta$

$\Leftrightarrow \tan^2\theta - 3\tan\theta + 2 = 0$

$X^2 - 3X + 2 = 0$

$\Delta = 9 - 4 \times 2 = 1$

So 2 solutions: $X = \dfrac{3-1}{2} = 1$ or $X = \dfrac{3+1}{2} = 2$

if $X = 1$ (i.e. $\tan\theta = 1$) $\quad \theta = \dfrac{\pi}{4} + n\pi$

So $\theta = \dfrac{\pi}{4} = 45°$ and $\theta = \dfrac{5\pi}{4} = 225°$

if $X = 2$ (i.e. $\tan\theta = 2$) there is no exact value

$\theta \simeq 63°26'$ and $\theta = 63°26' + 180° = 243°26'$

Four solutions: $45°,\ 63°26',\ 225°,\ 243°26'$

3 Solve: **(a)** $\cos 2x \cos \frac{\pi}{6} - \sin 2x \sin \frac{\pi}{6} = \frac{1}{2}, 0 \le x \le 2\pi$

This looks like $\cos A \cos B - \sin A \sin B = \cos(A+B)$

So $\Leftrightarrow \cos\left(2x + \frac{\pi}{6}\right) = \frac{1}{2} = \cos\frac{\pi}{3}$

General solution is $2x + \frac{\pi}{6} = \pm\frac{\pi}{3} + 2n\pi$

$\Leftrightarrow 2x = \pm\frac{\pi}{3} - \frac{\pi}{6} + 2n\pi$

$\Leftrightarrow x = \pm\frac{\pi}{6} - \frac{\pi}{12} + n\pi$

$n = 0$ gives $x = \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}$

and $x = -\frac{\pi}{6} - \frac{\pi}{12}$ outside of the interval $[0, 2\pi]$

$n = 1$ gives $x = \frac{\pi}{6} - \frac{\pi}{12} + \pi = \frac{13\pi}{12}$

and $x = -\frac{\pi}{6} - \frac{\pi}{12} + \pi = \frac{3\pi}{4}$

$n = 2$ gives $x = \frac{\pi}{6} - \frac{\pi}{12} + 2\pi$ outside of interval $[0, 2\pi]$

and $x = -\frac{\pi}{6} - \frac{\pi}{12} + 2\pi = \frac{7\pi}{4}$

no other solutions within the interval $[0, 2\pi]$

So four solutions: $\frac{\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{7\pi}{4}$

5 Solve $\tan\theta = \sin 2\theta$, $0 \le \theta \le 2\pi$.

$$\Leftrightarrow \frac{\sin\theta}{\cos\theta} = 2\sin\theta\cos\theta \Leftrightarrow \sin\theta - 2\sin\theta\cos^2\theta = 0$$

$$\Leftrightarrow \sin\theta\left[1 - 2\cos^2\theta\right] = 0$$

either $\sin\theta = 0$, i.e $\theta = 0$, $\theta = \pi$ or $\theta = 2\pi$

OR $1 - 2\cos^2\theta = 0 \Leftrightarrow \cos^2\theta = \frac{1}{2} \Leftrightarrow \cos\theta = \pm\frac{\sqrt{2}}{2}$

if $\cos\theta = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$ means $\theta = \pm\frac{\pi}{4} + 2n\pi$

$n = 0$ gives $\theta = \frac{\pi}{4}$ ($-\frac{\pi}{4}$ outside of $[0, 2\pi]$)

$n = 1$ gives $\theta = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

if $\cos\theta = -\frac{\sqrt{2}}{2} = \cos\frac{3\pi}{4}$ means $\theta = \pm\frac{3\pi}{4} + 2n\pi$

$n = 0$ gives $\theta = \frac{3\pi}{4}$

$n = 1$ gives $\theta = \frac{5\pi}{4}$

So 7 solutions $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$

6 Solve $\cos^2\theta = 2\cos^2\frac{\theta}{2}$, $0° \leq \theta \leq 360°$.

$$\Leftrightarrow \left[2\cos^2\left(\frac{\theta}{2}\right) - 1\right]^2 = 2\cos^2\left(\frac{\theta}{2}\right)$$

$$\Leftrightarrow 4\cos^4\left(\frac{\theta}{2}\right) - 4\cos^2\left(\frac{\theta}{2}\right) + 1 = 2\cos^2\left(\frac{\theta}{2}\right)$$

$$\Leftrightarrow 4\cos^4\left(\frac{\theta}{2}\right) - 6\cos^2\left(\frac{\theta}{2}\right) + 1 = 0 \Leftrightarrow 4X^2 - 6X + 1 = 0$$

with $X = \cos^2\left(\frac{\theta}{2}\right)$.

$\Delta = 36 - 4 \times 4 = 20$ 2 solutions $X_1 = \frac{6+\sqrt{20}}{8} = \frac{3+\sqrt{5}}{4}$

and $X_2 = \frac{3-\sqrt{5}}{4}$

if $X_1 = \frac{3+\sqrt{5}}{4} = \cos^2\left(\frac{\theta}{2}\right)$ impossible as $\frac{3+\sqrt{5}}{4} > 1$

if $X_2 = \frac{3-\sqrt{5}}{4} = \cos^2\left(\frac{\theta}{2}\right)$ so $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{3-\sqrt{5}}{4}} = \cos 64°5'$

or $= \cos 115°55'$

So $\frac{\theta}{2} = \pm 64°5' + n \times 360°$

$$\Leftrightarrow \theta = \pm 128°10' + n \times 720°$$

$n = 0$ gives $\theta = 128°10'$

OR $\frac{\theta}{2} = \pm 115°55' + n \times 360°$

$$\Leftrightarrow \theta = \pm 231°50' + n \times 720°$$

$n = 0$ gives $\theta = 231°50'$

7 Solve $\sin 3x \cos x - \cos 3x \sin x = \frac{\sqrt{3}}{2}$, $0 \le x \le 2\pi$.

$\sin 3x \cos x - \cos 3x \sin x = \sin(3x - x) = \sin 2x$

$\Leftrightarrow \quad \sin 2x = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}$

General solution is $2x = (-1)^n \times \frac{\pi}{3} + n\pi$

$\Leftrightarrow \quad x = (-1)^n \times \frac{\pi}{6} + n\frac{\pi}{2}$

$n = 0$ gives $x = \frac{\pi}{6}$

$n = 1$ gives $x = -\frac{\pi}{6} + \frac{\pi}{2} = \frac{\pi}{3}$

$n = 2$ gives $x = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$

$n = 3$ gives $x = -\frac{\pi}{6} + \frac{3\pi}{2} = \frac{4\pi}{3}$

no other solutions are within the interval $[0, 2\pi]$

9 Solve for $0 \le \theta \le \pi$, the equations:

(a) $\sin 4\theta \cos\theta = \sin 3\theta \cos 2\theta$

$\Leftrightarrow \quad \sin 4\theta \cos\theta - \sin 3\theta \cos 2\theta = 0$

$\Leftrightarrow \quad \frac{1}{2}\left[\sin(4\theta+\theta) + \sin(4\theta-\theta)\right] - \frac{1}{2}\left[\sin(3\theta+2\theta) + \sin(3\theta-2\theta)\right] = 0$

$\Leftrightarrow \quad \sin 3\theta - \sin\theta = 0 \quad \Leftrightarrow \quad \sin 3\theta = \sin\theta$

So $3\theta = (-1)^n \times \theta + n\pi$

$n = 0$ gives $3\theta = \theta \Leftrightarrow \theta = 0$

$n = 1$ gives $3\theta = -\theta + \pi \Leftrightarrow 4\theta = \pi \Leftrightarrow \theta = \frac{\pi}{4}$

$n = 2$ gives $3\theta = \theta + 2\pi \Leftrightarrow 2\theta = 2\pi \Leftrightarrow \theta = \pi$

$n = 3$ gives $3\theta = -\theta + 3\pi \Leftrightarrow 4\theta = 3\pi \Leftrightarrow \theta = \frac{3\pi}{4}$

$n = 4$ gives $3\theta = \theta + 4\pi \Leftrightarrow 2\theta = 4\pi \Leftrightarrow \theta = 2\pi$
which is outside of $[0, \pi]$

So 4 solutions $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi$

10 Solve for $0 \le x \le 2\pi$, the equations:

(a) $2\cos\left(x+\frac{\pi}{3}\right)\cos x = 1$ $\iff \cos\left(x+\frac{\pi}{3}\right)\cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

$$\iff \frac{1}{2}\left[\cos\left(x+\frac{\pi}{3}-x\right)+\cos\left(x+\frac{\pi}{3}+x\right)\right] = \frac{1}{2}$$

$$\iff \cos\left(2x+\frac{\pi}{3}\right)+\cos\left(\frac{\pi}{3}\right) = 1$$

$$\iff \cos\left(2x+\frac{\pi}{3}\right) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$$

General solution is $2x+\frac{\pi}{3} = \pm\frac{\pi}{3} + 2n\pi$

$$\iff 2x = \pm\frac{\pi}{3} - \frac{\pi}{3} + 2n\pi$$

$$\iff \boxed{x = \pm\frac{\pi}{6} - \frac{\pi}{6} + n\pi.}$$

$n=0$ gives $x = \frac{\pi}{6} - \frac{\pi}{6} = 0$

and $x = -\frac{\pi}{6} - \frac{\pi}{6} = -\frac{\pi}{3}$ (outside of $[0,2\pi]$)

$n=1$ gives $x = \frac{\pi}{6} + \frac{\pi}{6} + \pi = \pi$

and $x = -\frac{\pi}{6} - \frac{\pi}{6} + \pi = \frac{2\pi}{3}$

$n=2$ gives $x = \frac{\pi}{6} - \frac{\pi}{6} + 2\pi = 2\pi$

and $x = -\frac{\pi}{6} - \frac{\pi}{6} + 2\pi = \frac{5\pi}{3}$

other solutions are outside of interval $[0,2\pi]$

So 5 solutions $0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

11 Solve for $0 \le \theta \le \pi$, the equations:

(c) $\sin\left(\theta + \frac{\pi}{4}\right) + \sin\left(\theta + \frac{\pi}{12}\right) = 1$

$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

$\begin{cases} \alpha = A+B \\ \beta = A-B \end{cases}$ $\quad \begin{cases} A = \frac{\alpha+\beta}{2} \\ B = \frac{\alpha-\beta}{2} \end{cases}$ $\quad$ so $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\Leftrightarrow 2 \sin\left[\frac{\theta + \frac{\pi}{4} + \theta + \frac{\pi}{12}}{2}\right] \cos\left[\frac{\left(\theta+\frac{\pi}{4}\right) - \left(\theta + \frac{\pi}{12}\right)}{2}\right] = 1$

$\Leftrightarrow \sin\left[\theta + \frac{\pi}{6}\right] \cos\left[\frac{\pi}{12}\right] = \frac{1}{2}$

$\Leftrightarrow \sin\left[\theta + 30°\right] = \frac{1/2}{\cos 15°} \simeq 0.5176 = \sin 31°10'$

General solution is $\theta + 30° = 31°10' \times (-1)^n + 180 n$

$\theta = (-1)^n \times 31°10' - 30° + 180 n$

$n = 0$ gives $\theta = 1°10'$

$n = 1$ gives $\theta = 118°50'$

$n = 2$ gives $\theta =$ outside of $[0, 180°]$

12 Solve for $0 \le x \le 2\pi$, the equations:

(a) $\sin 2x - \sin x = \cos 2x - \cos x$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

$\begin{cases} \alpha = A+B \\ \beta = A-B \end{cases}$ so $A = \dfrac{\alpha+\beta}{2}$ and $B = \dfrac{\alpha-\beta}{2}$

so $\sin\alpha - \sin\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

$\begin{cases} \alpha = A-B \\ \beta = A+B \end{cases}$ so $A = \dfrac{\alpha+\beta}{2}$ and $B = \dfrac{\beta-\alpha}{2}$

so $\cos\alpha - \cos\beta = 2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\beta-\alpha}{2}\right)$

So the equation becomes: $2\cos\left(\dfrac{3x}{2}\right)\sin\left(\dfrac{x}{2}\right) = 2\sin\left(\dfrac{3x}{2}\right)\sin\left(-\dfrac{x}{2}\right)$

$\Leftrightarrow \quad \cos\left(\dfrac{3x}{2}\right)\sin\left(\dfrac{x}{2}\right) = \sin\left(\dfrac{3x}{2}\right) \times (-1) \times \sin\left(\dfrac{x}{2}\right)$

$\Leftrightarrow \quad \cos\left(\dfrac{3x}{2}\right)\sin\left(\dfrac{x}{2}\right) + \sin\left(\dfrac{3x}{2}\right)\sin\left(\dfrac{x}{2}\right) = 0$

$\Leftrightarrow \quad \sin\left(\dfrac{x}{2}\right)\left[\cos\left(\dfrac{3x}{2}\right) + \sin\left(\dfrac{3x}{2}\right)\right] = 0$

So either $\sin\left(\dfrac{x}{2}\right) = 0$ i.e $\dfrac{x}{2} = n\pi$ $\quad x = 0$ or $x = 2\pi$

or $\cos\left(\dfrac{3x}{2}\right) + \sin\left(\dfrac{3x}{2}\right) = 0 \Leftrightarrow \tan\left(\dfrac{3x}{2}\right) = -1 = \tan\left(\dfrac{3\pi}{4}\right)$

General solution is $\dfrac{3x}{2} = \dfrac{3\pi}{4} + n\pi \Leftrightarrow x = \dfrac{\pi}{2} + \dfrac{2n\pi}{3}$

$n = 0$ gives $x = \dfrac{\pi}{2}$ $\qquad n = 1$ gives $x = \dfrac{7\pi}{6}$

$n = 2$ gives $x = \dfrac{11\pi}{6}$ $\qquad$ no other solutions.

So 5 solutions: $0, \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, 2\pi$