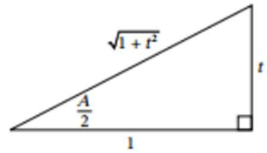
THE SUBSTITUTION $t = \tan\frac{A}{2}$

The substitution $t = \tan\frac{A}{2}$ enables you to express sin A and cos A in terms of t, which then enables you to express any rational function of sin A and cos A as a rational algebraic function of t. This then allows you to use standard techniques of integration such as partial fractions or integration by parts, covered in Chapter 5.

If $t = \tan\frac{A}{2}$, then it follows from the right-angled triangle that:



$$\sin\frac{A}{2} = \frac{t}{\sqrt{1+t^2}}$$

$$\cos\frac{A}{2} = \frac{1}{\sqrt{1+t^2}}$$

Now $\quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\times\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$

and $\quad \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$

As $t = \tan\frac{A}{2}$: $\quad \frac{dt}{dA} = \frac{1}{2}\sec^2\frac{A}{2} = \frac{1+t^2}{2} \qquad \therefore \frac{dA}{dt} = \frac{2}{1+t^2}$

Summary—*t* formulae

- $t = \tan\frac{A}{2}$
- $\sin A = \frac{2t}{1+t^2}$
- $\cos A = \frac{1-t^2}{1+t^2}$
- $\frac{dA}{dt} = \frac{2}{1+t^2}$

Example 12

Find: $\displaystyle\int\frac{dx}{1+\cos x}$

Solution

Let $t = \tan\frac{x}{2}$ so that: $\quad 1+\cos x = 1 + \frac{1-t^2}{1+t^2} = \frac{1+t^2+1-t^2}{1+t^2} = \frac{2}{1+t^2}$

Use $dx = \frac{2}{1+t^2}dt$: $\quad \displaystyle\int\frac{dx}{1+\cos x} = \int\frac{1+t^2}{2}\times\frac{2}{1+t^2}dt$

$$= \int dt$$

$$= t + C$$

$$= \tan\frac{x}{2} + C$$

Alternatively:

From the double-angle results: $\quad 2\cos^2\frac{x}{2} = 1+\cos x$

$$\therefore \int\frac{dx}{1+\cos x} = \frac{1}{2}\int\frac{dx}{\cos^2\frac{x}{2}} = \frac{1}{2}\int\sec^2\frac{x}{2}dx = \tan\frac{x}{2} + C$$

Example 13

Evaluate: $\int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin x}$

Solution

Let $t = \tan\frac{x}{2}$ so that: $2-\sin x = 2 - \frac{2t}{1+t^2} = \frac{2(1-t+t^2)}{1+t^2}$

Use $dx = \frac{2}{1+t^2}dt$: $\int \frac{dx}{2-\sin x} = \int \frac{1+t^2}{2(1-t+t^2)} \times \frac{2}{1+t^2}dt$

$$= \int \frac{dt}{t^2-t+1}$$

Complete the square: $= \int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \frac{3}{4}}$

$$= \int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

But $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$: $= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

As $t = \tan\frac{x}{2}$: $\int \frac{dx}{2-\sin x} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}-1}{\sqrt{3}}\right) + C$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin x} = \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}-1}{\sqrt{3}}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2-1}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right)$$

$$= \frac{2\pi}{3\sqrt{3}}$$