INTRODUCTION TO DIFFERENTIATION - CHAPTER REVIEW

1 Find the following limits. **(a)** $\lim_{x \to \frac{1}{2}} \frac{1-4x^2}{1-2x}$ **(b)** $\lim_{x \to 3} \frac{x^3-27}{x-3}$

a) $\lim_{x \to 1/2} \frac{1-4x^2}{1-2x} = \lim_{x \to 1/2} \frac{(1-2x)(1+2x)}{(1-2x)} = \lim_{x \to 1/2} (1+2x) = 2$

b) $\lim_{x \to 3} \frac{x^3-27}{x-3} = \lim_{x \to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)}$

$= \lim_{x \to 3} x^2+3x+9$

$= 3^2 + 3 \times 3 + 9$

$= 27$

2 Evaluate: **(a)** $\lim_{h \to 0} \frac{2x^2h+3h}{h}$ **(b)** $\lim_{h \to 0} \frac{(2+h)^2-4}{h}$ **(c)** $\lim_{h \to 0} \frac{(1+h)^3-1}{h}$

a) $\lim_{h \to 0} \frac{2x^2h+3h}{h} = \lim_{h \to 0} 2x^2+3 = 2x^2+3$

b) $\lim_{h \to 0} \frac{(2+h)^2-4}{h} = \lim_{h \to 0} \frac{4+4h+h^2-4}{h} = \lim_{h \to 0} (4+h) = 4$

c) $\lim_{h \to 0} \frac{(1+h)^3-1}{h} = \lim_{h \to 0} \frac{1+3h+3h^2+h^3-1}{h}$

$= \lim_{h \to 0} \frac{3h+3h^2+h^3}{h}$

$= \lim_{h \to 0} 3+3h+h^2$

$= 3$

3 Find $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$, for $f(x)=2x^2-3x$.

$$\frac{f(x+h)-f(x)}{h} = \frac{[2(x+h)^2-3(x+h)]-[2x^2-3x]}{h}$$

$$= \frac{[2(x^2+2hx+h^2)-3x-3h]-2x^2+3x}{h}$$

$$= \frac{[2x^2+4hx+2h^2-3x-3h]-2x^2+3x}{h}$$

$$= \frac{\cancel{2x^2}+4hx+2h^2\cancel{-3x}-3h\cancel{-2x^2}\cancel{+3x}}{h}$$

$$= \frac{4hx+2h^2-3h}{h}$$

$$= 4x+2h-3$$

4 For $f(x) = x^2 + 6x + 8$, find:

(a) $f(2)$ **(b)** $f'(2)$ **(c)** $f'(c)$ **(d)** the value of c for which $f'(c) = -2$

a) $f(2) = 2^2 + 6 \times 2 + 8 = 4 + 12 + 8 = 24$

b) $f'(x) = 2x + 6$

So $f'(2) = 2 \times 2 + 6 = 10$

c) $f'(c) = 2c + 6$

d) for $f'(c)$ to be equal to (-2), we must have:

$2c + 6 = -2$

$\Leftrightarrow \quad 2c = -8$

$\Leftrightarrow \quad \boxed{c = -4}$

5 Find $f'(x)$ for $f(x)=\sqrt{2x-1}$. $= (2x-1)^{1/2}$

We use the chain rule:

$$f'(x) = \frac{1}{2}(2x-1)^{1/2-1} \times 2$$

So $f'(x) = (2x-1)^{-1/2}$

$$f'(x) = \frac{1}{(2x-1)^{1/2}} = \frac{1}{\sqrt{2x-1}}$$

6 Given $y=(x^2-4)(3x^2-2x+1)^5$, find $\frac{dy}{dx}$. We use the product rule. and the chain rule.

$f(x) = u(x) \times v(x)$

$u(x) = x^2-4$ $\qquad u'(x) = 2x$

$v(x) = (3x^2-2x+1)^5$ $\qquad v'(x) = 5(3x^2-2x+1)^4 \times (6x-2)$

So $f'(x) = 2x\,(3x^2-2x+1)^5 + 5(3x^2-2x+1)^4(6x-2)(x^2-4)$

$$\frac{df}{dx} = (3x^2-2x+1)^4\left[2x(3x^2-2x+1)+5(6x-2)(x^2-4)\right]$$

$$\frac{df}{dx} = (3x^2-2x+1)^4\left[6x^3-4x^2+2x+5(6x^3-24x-2x^2+8)\right]$$

$$\frac{df}{dx} = (3x^2-2x+1)^4\left[36x^3-14x^2-118x+40\right]$$

$$\frac{df}{dx} = (3x^2-2x+1)^4 \times 2\left[18x^3-7x^2-59x+20\right]$$

7 Find the derivative of each function.

(a) $y=(x-2)^3$ **(b)** $f(x)=(x^2+x^3)^5$ **(c)** $v=\sqrt{25-t^2}$ $= (25-t^2)^{1/2}$

(d) $y=(x+x^{-1})^2$ **(e)** $g(x)=\frac{1}{(x+4)^2}$ **(f)** $y=\frac{x-2}{x}$

a) $f'(x) = 3(x-2)^2 \times 1$

b) $f'(x) = 5(x^2+x^3)^4 \times (2x+3x^2)$

c) $\frac{dv}{dt} = \frac{1}{2}(25-t^2)^{1/2-1} \times (-2t) = -t(25-t^2)^{-1/2} = -\frac{t}{\sqrt{25-t^2}}$

d) $\frac{dy}{dx} = 2(x+x^{-1}) \times (1 - 1 \times x^{-1-1}) = 2(x+x^{-1})(1-x^{-2})$

e) $g(x) = (x+4)^{-2}$ so $g'(x) = -2(x+4)^{-2-1} \times 1$

$g'(x) = -2(x+4)^{-3} = \frac{-2}{(x+4)^3}$

f) $y = 1 - \frac{2}{x} = 1 - 2x^{-1}$

So $\frac{dy}{dx} = 0 - 2 \times (-1)\, x^{-1-1} = 2x^{-2}$

$\frac{dy}{dx} = \frac{2}{x^2}$

7 Find the derivative of each function.

(j) $u=\dfrac{2m-7}{2m+3}$ (k) $y=\dfrac{1+x^3}{x^2}$ (l) $h(t)=(t-3)\sqrt{t-3}$

j) we use the quotient rule.

$u(m)=\dfrac{2m-7}{2m+3}=\dfrac{f(m)}{g(m)}$ $f(m)=2m-7$ $f'(m)=2$

$g(m)=2m+3$ $g'(m)=2$

$u'(m)=\dfrac{2(2m+3)-2(2m-7)}{(2m+3)^2}$.

$u'(m)=\dfrac{4m+6-4m+14}{(2m+3)^2}=\dfrac{20}{(2m+3)^2}$

k) $y=x^{-2}+x$ so $\dfrac{dy}{dx}=-2x^{-2-1}+1=\dfrac{-2}{x^3}+1$

l) $h(t)=(t-3)^{3/2}$

So $h'(t)=\dfrac{3}{2}(t-3)^{3/2-1}\times 1=\dfrac{3}{2}(t-3)^{1/2}=\dfrac{3}{2}\sqrt{t-3}$

9 Find the equation of the tangent to the parabola $y = 4x - x^2$ at the point where the gradient is -2.

To find the tangent, we need to find 1) its gradient 2) one point.

1) is (-2) where $\frac{dy}{dx} = 4 - 2x$

so $4 - 2x = -2 \iff 2x = 6 \iff x = 3$

i.e. the gradient at $x = 3$ is (-2)

2) at $x = 3$ $f(3) = 4 \times 3 - 3^2 = 12 - 9 = 3$

So the tangent goes through the point $(3,3)$.

∴ the equation of the tangent is $y - y_0 = m(x - x_0)$

or $y - 3 = -2(x - 3) \iff y = -2x + 6 + 3$

$y = -2x + 9$

10 Find the equation of the tangents to the curve $y = 2x^2(4 - x)$ at the point where the curve intersects the x-axis.

The curve intersects the x-axis at $x = 0$ and $x = 4$

$[f(0) = 0$ and $f(4) = 0]$

The derivative is $\frac{dy}{dx} = 16x - 2 \times 3x^2 = 16x - 6x^2$
$= 2x[8 - 3x]$

So $f'(0) = 0$

$y - y_0 = m(x - x_0)$

$y - 0 = 0(x - 0)$

$\boxed{y = 0}$

and $f'(4) = 2 \times 4[8 - 3 \times 4] = -32$

$y - y_0 = m(x - x_0)$

$y - 0 = -32(x - 4)$

$\boxed{y = -32x + 128}$

12 A particle is moving along the x-axis and is initially at the origin. Its velocity v metres per second at time t seconds is given by $v = \frac{2t}{9+t^2}$.

(a) What is the initial velocity of the particle?
(b) Find an expression for the acceleration of the particle.
(c) When is the acceleration zero?
(d) What is the maximum velocity attained by the particle and when does it occur?

a) at $t = 0$ $\quad v(0) = \frac{2 \times 0}{9 + 0^2} = 0 \ ms^{-1}$

b) $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{2t}{9+t^2}\right)$ we use quotient rule.

$u(t) = 2t \qquad u'(t) = 2$

$w(t) = 9 + t^2 \qquad w'(t) = 2t$

So $a = \frac{2(9+t^2) - 2t \times 2t}{(9+t^2)^2} = \frac{18 + 2t^2 - 4t^2}{(9+t^2)^2} = \frac{18 - 2t^2}{(9+t^2)^2}$

c) $a = 0$ when $18 - 2t^2 = 0$, i.e. $t^2 = 9$ so $t = 3$
(the negative value is not possible for t)

d) Maximum velocity occurs when $a = 0$, i.e. when $t = 3$

For $t = 3$ $\quad v(3) = \frac{2 \times 3}{(9 + 3^2)} = \frac{6}{18} = \frac{1}{3} \ ms^{-1}$

1 Differentiate:

(a) $e^{x^2}+2$ **(b)** $\left(e^x+x^2\right)^4$ **(c)** e^x+ex

a) we use the Chain rule to differentiate e^{x^2}

$f(x)=e^{x^2}+2$

$f'(x) = e^{x^2} \times 2x = 2x\,e^{x^2}$

b) Chain rule $f(x) = (e^x+x^2)^4$

$f'(x) = 4\,(e^x+x^2)^3 \times [e^x+2x]$

c) $f(x) = e^x + ex$

$f'(x) = e^x + e$

5 In statistics, the normal probability density function is given by $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. Find $f'(0)$.

$$f'(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \times \left(-\frac{2x}{2}\right) = -\frac{x}{\sqrt{2\pi}}\, e^{-x^2/2}$$

$$\text{So } f'(0) = -\frac{0}{\sqrt{2\pi}}\, e^{-0^2/2} = 0$$

2 Differentiate (a) $(x^2+2x)e^x$ **(c)** $2^x+3^x+4^x$ **(e)** $(x^2+3x)e^{-3x}$

a) $f(x) = (x^2+2x)e^x$ we use the product rule.

$u(x) = x^2 + 2x$ $u'(x) = 2x + 2$

$v(x) = e^x$ $v'(x) = e^x$

$f'(x) = (2x+2)e^x + (x^2+2x)e^x = e^x[x^2+4x+2]$

c) $f(x) = 2^x + 3^x + 4^x = e^{x\ln 2} + e^{x\ln 3} + e^{x\ln 4}$

$f'(x) = \ln 2\, e^{x\ln 2} + \ln 3\, e^{x\ln 3} + \ln 4\, e^{x\ln 4}$

$f'(x) = \ln 2 \times 2^x + \ln 3 \times 3^x + \ln 4 \times 4^x$

e) $f(x) = (x^2+3x)e^{-3x}$

$u(x) = x^2 + 3x$ $u'(x) = 2x + 3$

$v(x) = e^{-3x}$ $v'(x) = -3e^{-3x}$

$f'(x) = (2x+3)e^{-3x} - 3e^{-3x}(x^2+3x)$

$f'(x) = e^{-3x}[-3x^2 - 9x + 2x + 3]$

$f'(x) = e^{-3x}[-3x^2 - 7x + 3]$