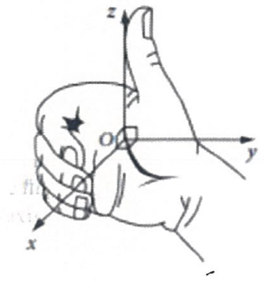
VECTORS IN THREE DIMENSIONS

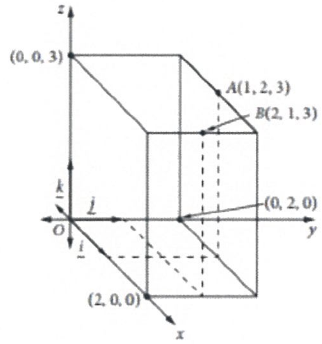
The work that follows requires you to be able to visualise objects in three-dimensional space. You need to be able to draw cubes and rectangular prisms with some of the edges forming the axes of the Cartesian system.

To describe the position of a point in three-dimensional space, you need three coordinates. The Cartesian system of two coordinates, X and Y, is extended by means of a third axis, OZ, which is perpendicular to the plane OXY. The positive direction of OZ is towards the top of the page, the positive direction of OX is out of the page and the positive direction of OY is horizontally to the right.

This is shown in the diagram on the right. It is called a right-hand system of axes. Here, the fingers of the right hand point in the direction from the positive x-axis to the positive y-axis, so that the thumb points upwards in the direction of the positive z-axis.



In the following diagram, the x–y plane is horizontal, the x–z and y–z planes are vertical. In this $OXYZ$ system, the points $A(1, 2, 3)$ and $B(2, 1, 3)$ are shown.



The vector $\overrightarrow{OA}$ may be written as $\overrightarrow{OA} = (1, 2, 3)$, $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ or $\overrightarrow{OA} = \underset{\sim}{a}$, as well as in component form.

The components of a vector in three dimensions use $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ as the unit vectors parallel to the x, y and z axes respectively.

Thus $\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\overrightarrow{OB} = \underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$.

Using your knowledge of two-dimensional vectors, it should seem reasonable to find $\overrightarrow{AB}$.

$$\begin{aligned} \overrightarrow{AB} &= \underset{\sim}{b} - \underset{\sim}{a} = \left(2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}\right) - \left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) \\ &= (2-1)\underset{\sim}{i} + (1-2)\underset{\sim}{j} + (3-3)\underset{\sim}{k} \\ &= \underset{\sim}{i} - \underset{\sim}{j} \end{aligned}$$

$\overrightarrow{AB}$ has no $\underset{\sim}{k}$ component, so it lies in a plane that is parallel to the x–y plane. It is the plane given by the equation $z = 3$.

In particular, points in the x–y plane are of the form $(x, y, 0)$, points in the y–z plane are of the form $(0, y, z)$ and points in the x–z plane are of the form $(x, 0, z)$.

In two-dimensional space, the coordinate axes divide the plane into four regions or quadrants. In three-dimensional space, the coordinate axes divide the space into eight regions or octants. The sign of the coordinates indicates in which octant the point is located.

Example 1

Given $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, $\underset{\sim}{b} = (-1, 3, 2)$ and $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$, find each of the following vectors, expressing your answer in component form.

(a) $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ **(b)** $\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c}$ **(c)** $\underset{\sim}{a} - \underset{\sim}{b} - \underset{\sim}{c}$ **(d)** $4\underset{\sim}{a}$ **(e)** $\underset{\sim}{a} - 2\underset{\sim}{b} + 3\underset{\sim}{c}$

Solution

The answers are to be given in component form, so rewrite each vector in component form.

$\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$.

(a) $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} + \left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) + \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$= 2\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$

(b) $\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} - \left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) + \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} + \underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k} + \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$= 4\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

(c) $\underset{\sim}{a} - \underset{\sim}{b} - \underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} - \left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) - \left(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$

$= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} + \underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k} - \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$= 2\underset{\sim}{i} - 6\underset{\sim}{j} + 5\underset{\sim}{k}$

(d) $4\underset{\sim}{a} = 4\left(2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}\right)$

$= 8\underset{\sim}{i} - 4\underset{\sim}{j} + 16\underset{\sim}{k}$

(e) $\underset{\sim}{a} - 2\underset{\sim}{b} + 3\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} - 2\left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) + 3\left(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$

$= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} + 2\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k} + 3\underset{\sim}{i} + 6\underset{\sim}{j} - 9\underset{\sim}{k}$

$= 7\underset{\sim}{i} - \underset{\sim}{j} - 9\underset{\sim}{k}$

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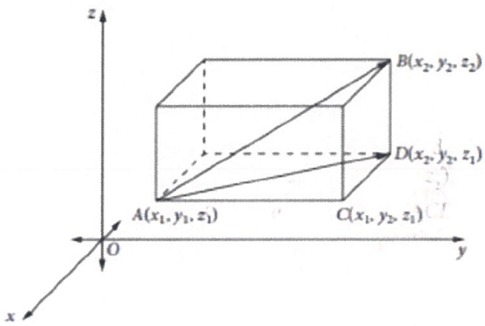
Magnitude of vectors in three-dimensional space

To obtain the magnitude of vectors in three-dimensional space you use an extension of Pythagoras' theorem. This is similar to the method used to find the length of a diagonal of a cuboid in solid geometry.

Suppose you wish to find the length of the line segment joining $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$, as shown in the diagram.

Construct the planes ACD and CDB that are parallel to the x–y and x–z planes respectively. Then C, which has the same x and z coordinates as A, will be $C(x_1, y_2, z_1)$, while D, which has the same x and y coordinates as B, will be $D(x_2, y_2, z_1)$.



Now $\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{DB}\right|^2$

$= \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{CD}\right|^2 + \left|\overrightarrow{DB}\right|^2$

And $\left|\overrightarrow{AC}\right| = |y_2 - y_1|$

$\left|\overrightarrow{CD}\right| = |x_2 - x_1|$

$\left|\overrightarrow{DB}\right| = |z_2 - z_1|$

Hence, $\left|\overrightarrow{AB}\right|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ and thus $\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Referring to diagram on page 79:

$\left|\overrightarrow{OA}\right| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$\left|\overrightarrow{OB}\right| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$

$\left|\overrightarrow{AB}\right| = \sqrt{(2-1)^2 + (1-2)^2 + (3-3)^2} = \sqrt{2}$

Unit vectors in three-dimensional space

Given $\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\overrightarrow{OB} = \underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, then since $\left|\overrightarrow{OA}\right| = \sqrt{14}$ and $\left|\overrightarrow{OB}\right| = \sqrt{14}$, the corresponding unit vectors can be found using the fact that $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

$\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{14}}\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) = \frac{\sqrt{14}}{14}\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$ and $\hat{\underset{\sim}{b}} = \frac{1}{\sqrt{14}}\left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right) = \frac{\sqrt{14}}{14}\left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$.

Thus $\hat{\underset{\sim}{a}} = \frac{\sqrt{14}}{14}\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$ is the unit vector parallel to $\overrightarrow{OA}$ and $\hat{\underset{\sim}{b}} = \frac{\sqrt{14}}{14}\left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$ is the unit vector parallel to $\overrightarrow{OB}$.

From the given vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ you found that $\overrightarrow{AB} = \underset{\sim}{i} - \underset{\sim}{j}$ and that $\left|\overrightarrow{AB}\right| = \sqrt{2}$ so the unit vector parallel to $\overrightarrow{AB}$ is given by $\frac{\sqrt{2}}{2}\left(\underset{\sim}{i} - \underset{\sim}{j}\right)$. Since this vector does not contain a $\underset{\sim}{k}$ component, it lies in a plane parallel to the x–y plane.

Example 2

(a) On a set of Cartesian axes, mark the terminal points $A(1, 3, 4)$, $B(-1, 3, 2)$ of vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$.

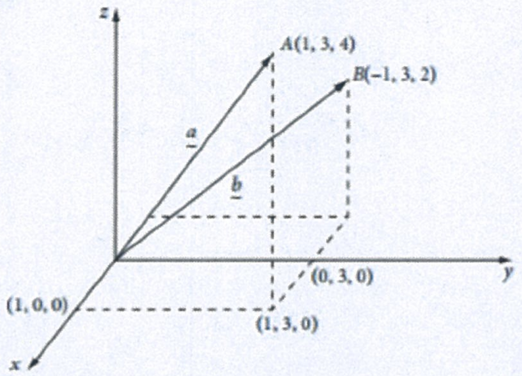
(b) Write the vectors $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, in terms of $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$. Similarly, write an expression for $\underset{\sim}{e} = \underset{\sim}{b} - \underset{\sim}{a}$.

(c) Find the magnitudes of vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{e}$.

(d) Find unit vectors in the direction of the vectors given in part **(c)**.

Solution

(a)



Note: **(i)** $\underset{\sim}{a}$ points out from the page upwards towards the reader.

(ii) $\underset{\sim}{b}$ points into the page.

(b) $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}, \underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}, \underset{\sim}{e} = \overrightarrow{AB} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} - (\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}) = -2\underset{\sim}{i} - 2\underset{\sim}{k}.$

(c) $|\underset{\sim}{a}| = \sqrt{1+9+16} = \sqrt{26}, |\underset{\sim}{b}| = \sqrt{1+9+4} = \sqrt{14}, |\underset{\sim}{e}| = \sqrt{4+4} = 2\sqrt{2}.$

(d) $\hat{\underset{\sim}{a}} = \frac{\sqrt{26}}{26}\left(\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}\right), \hat{\underset{\sim}{b}} = \frac{\sqrt{14}}{14}\left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right), \hat{\underset{\sim}{e}} = \frac{\sqrt{2}}{4}\left(-\underset{\sim}{i} - \underset{\sim}{k}\right).$

Position vectors in three dimensions

A **vector** defines the position of one point relative to another point. When the reference point is the origin, the vector is simply called a 'position vector'. When the reference point is not the origin, the term **relative position vector** is used.

Given the position vector of A is $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$ and the position vector of B is $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, the position vector of B *relative to* A (i.e. B as seen from A) is

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{b} - \underset{\sim}{a} \\ &= -2\underset{\sim}{i} - 2\underset{\sim}{k}.\end{aligned}$$

The position vector of A relative to B (i.e. A as seen from B) is $\overrightarrow{BA} = 2\underset{\sim}{i} + 2\underset{\sim}{k}$.

Note that the position vector of B relative to A = position vector of B − position vector of A.

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Algebra of vectors expressed in component form

(i) Equality: Two vectors are equal if and only if the corresponding components are equal;
i.e. $a_1\underset{\sim}{i} + b_1\underset{\sim}{j} + c_1\underset{\sim}{k} = a_2\underset{\sim}{i} + b_2\underset{\sim}{j} + c_2\underset{\sim}{k}$ if and only if $a_1 = a_2, b_1 = b_2, c_1 = c_2$.
(This is true because the $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ representation of a vector is unique.)

(ii) The components of the sum (difference) of two vectors are equal to the sum (difference) of the corresponding components of the vectors;
e.g. If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$, then $\underset{\sim}{a} \pm \underset{\sim}{b} = (x_1 \pm x_2)\underset{\sim}{i} + (y_1 \pm y_2)\underset{\sim}{j} + (z_1 \pm z_2)\underset{\sim}{k}$.

(iii) The components of a scalar multiple of a vector are equal to the scalar multiple of the corresponding components of the vector;
e.g. If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ and $\lambda \in R$, then $\lambda\underset{\sim}{a} = (\lambda x)\underset{\sim}{i} + (\lambda y)\underset{\sim}{j} + (\lambda z)\underset{\sim}{k}$.

Example 3

Given $A(6, 4, 8)$ and $B(10, 6, 8)$ show that $\overrightarrow{AB}$ is parallel to the x–y plane.

Solution

$$\overline{AB} = (10-6)\underset{\sim}{i} + (6-4)\underset{\sim}{j} + (8-8)\underset{\sim}{k}$$
$$= 4\underset{\sim}{i} + 2\underset{\sim}{j}$$

Since $\overline{AB}$ has no $\underset{\sim}{k}$ component, it is parallel to the x–y plane.

Example 4

Determine whether the points $A(0, 2, 2)$, $B(4, 10, 18)$, $C(6, 14, 26)$ are collinear.

Solution

$$\overrightarrow{AB} = (4-0)\underset{\sim}{i} + (10-2)\underset{\sim}{j} + (18-2)\underset{\sim}{k}$$
$$= 4\underset{\sim}{i} + 8\underset{\sim}{j} + 16\underset{\sim}{k}$$
$$= 4(\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})$$

$$\overrightarrow{BC} = (6-4)\underset{\sim}{i} + (14-10)\underset{\sim}{j} + (26-18)\underset{\sim}{k}$$
$$= 2\underset{\sim}{i} + 4\underset{\sim}{j} + 8\underset{\sim}{k}$$
$$= 2(\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})$$

Hence $\overrightarrow{AB} = 2\overrightarrow{BC}$.
Thus $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel and have point B in common.
Hence ABC is a straight line and the points A, B and C are collinear.

Example 5

Find the distance of the point $P(2, 3, 5)$:

(a) from the y–z plane

(b) from the x-axis.

Solution

(a) The line segment from P normal to the y–z plane intersects the y–z plane at $Q(0, 3, 5)$.
Hence $|\overrightarrow{PQ}| = |0-2| = 2$.

(b) The line segment from P perpendicular to the x-axis meets the x-axis at $R(2, 0, 0)$.
Thus $|\overrightarrow{PR}| = \sqrt{0^2 + 3^2 + 5^2} = \sqrt{34}$.