USING DERIVATIVES IN MOTION IN A STRAIGHT LINE

Displacement

Displacement is defined as the position relative to a starting point. It can be positive or negative. Displacement does not necessarily represent the total distance travelled.

Unlike displacement, distance is always a positive quantity.

Velocity

Velocity is defined as the rate of change of position (i.e. of displacement) with respect to time, or as the time rate of change of position in a given direction.

$$v(t) = f'(t) = \frac{dx}{dy} = \dot{x} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Velocity can be positive or negative, depending on the direction of travel.

Speed is the magnitude of the velocity and is always positive.

Acceleration

Acceleration is defined as the rate of change of velocity with respect to time. Acceleration, like velocity, can be positive or negative. Positive acceleration indicates that the velocity is increasing, while negative acceleration indicates that the velocity is decreasing, which is often called deceleration or retardation.

(Note that 'increasing velocity' is not necessarily 'faster speed'; it only means acceleration in the direction of positive displacement.)

If you denote the velocity by $v(t)$, then the average acceleration over the interval from t to $(t + h)$ is $\frac{v(t+h) - v(t)}{h}$.

The instantaneous acceleration at time t is defined by $\lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$. It may be denoted by

$$v'(t),\ a(t),\ f''(t),\ \frac{dv}{dt},\ \frac{d^2x}{dt^2}, \text{ or } \ddot{x} : a(t) = v'(t) = \frac{d^2x}{dt^2} = \ddot{x} = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

Summary of important terms

'initially': $t = 0$ 'at the origin': $x = 0$

'at rest': $v = 0$ 'velocity is constant': $a = 0$

Units and symbols

Physical quantity	Unit	Symbol
Time	s	t
Displacement	cm, m	x (or s in physics)
Velocity	cm s^{-1}, m s^{-1}	v, $\frac{dx}{dt}$, $\dot{x}$
Acceleration	cm s^{-2}, m s^{-2}	a, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $\ddot{x}$

Note that 's' is the abbreviation for second, 'cm' for centimetre and 'm' for metre.
Constant acceleration due to gravity = $9.8\ \text{m s}^{-2}$.

Example 18

A ball is projected vertically upwards from the top of a building 30 metres high. The equation for its motion is given by $x = 30 + 25t - 5t^2$, where x is the displacement in metres above the top of the building and t is in seconds.

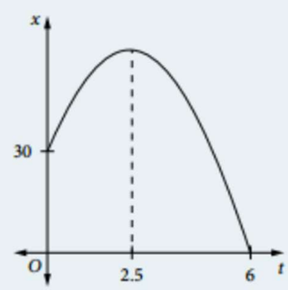
(a) Graph the displacement function.
(b) Find the velocity as a function of time.
(c) What is the initial velocity of the ball?
(d) The ball reaches its greatest height when $\frac{dx}{dt} = 0$. When does it reach its greatest height and how high above the ground is it then?
(e) How long will it take for the ball to hit the ground?
(f) What is the ball's speed when it hits the ground?
(g) Find the expression for the acceleration of the ball.

Solution

(a) $x = 30 + 25t - 5t^2$

The graph will be a parabola so the axis of symmetry is given by $t = \frac{-25}{2\times(-5)} = 2.5$

t	0	1	2	2.5	3	4	5	6
x	30	50	60	61.25	60	50	30	0



(b) $x = 30 + 25t - 5t^2$: $v = \frac{dx}{dt} = 25 - 10t$

(c) Initial velocity at $t = 0$: $v = 25\ \text{m s}^{-1}$

(d) $\frac{dx}{dt} = 0$: $25 - 10t = 0$

$t = 2.5$ seconds (This could have been obtained from the graph).

$t = 2.5$: $x = 61.25$ m

The highest point is 61.25 metres above the ground, which the ball reaches after 2.5 seconds.

(e) The ball reaches the ground when $x = 0$: using the graph the answer is 6 seconds.

Using the displacement function: $30 + 25t - 5t^2 = 0$

$$5(6 + 5t - t^2) = 0$$
$$(6 - t)(1 + t) = 0$$

As $t \geq 0$, $t = 6$.

The ball hits the ground after 6 seconds.

(f) $t = 6$: $v = 25 - 60 = -35$

The ball hits the ground with a speed of $35\ \text{m s}^{-1}$.
As the initial upwards velocity is positive, the velocity when it hits the ground is negative as the ball is moving downwards.

(g) $a = \frac{dv}{dt} = -10\ \text{m s}^{-2}$

This means that the acceleration is acting in the opposite direction to the initial upwards velocity.