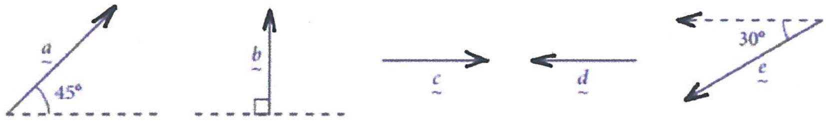
SCALAR PRODUCT OF VECTORS

1 Consider the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{d}$ and $\underset{\sim}{e}$ as shown.

Find the angle between the following pairs of vectors.



(a) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (b) $\underset{\sim}{a}$ and $\underset{\sim}{c}$ (c) $\underset{\sim}{a}$ and $\underset{\sim}{d}$ (d) $\underset{\sim}{a}$ and $\underset{\sim}{e}$ (e) $\underset{\sim}{b}$ and $\underset{\sim}{c}$
(f) $\underset{\sim}{b}$ and $\underset{\sim}{d}$ (g) $\underset{\sim}{b}$ and $\underset{\sim}{e}$ (h) $\underset{\sim}{c}$ and $\underset{\sim}{d}$ (i) $\underset{\sim}{c}$ and $\underset{\sim}{e}$ (j) $\underset{\sim}{d}$ and $\underset{\sim}{e}$

a) $45°$

b) $-45°$

c) $135°$

d) $135 + 30 = 165°$

e) $270°$

f) $90°$

g) $120°$

h) $180°$

i) $210°$

j) $30°$

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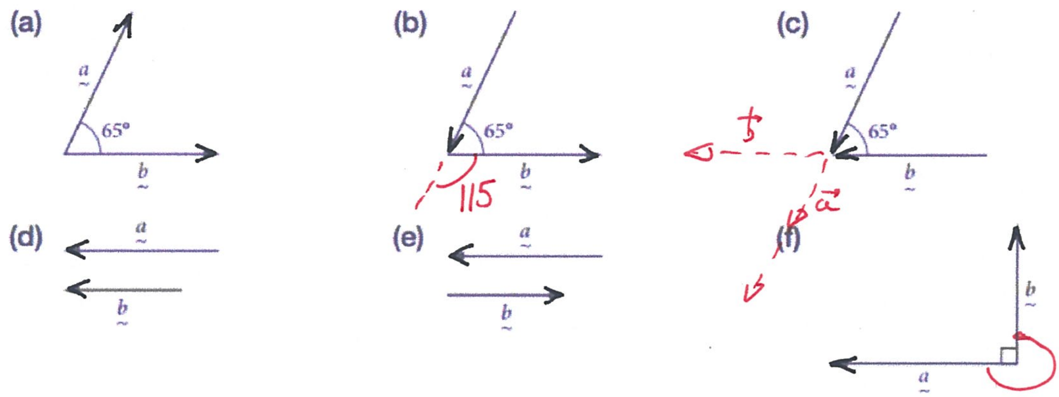
2 Given $|\underset{\sim}{a}| = 8$ and $|\underset{\sim}{b}| = 7$, find the scalar product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ for each of the following, correct to two decimal places where necessary.



a) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}| \cos 65 = 8\times 7\times \cos 65 = 23.67$

b) $\vec{a}\cdot\vec{b} = 8\times 7 \cos(25+90) = -23.67$

c) $\vec{a}\cdot\vec{b} = 8\times 7 \cos(360-65) = 23.67$

d) $\vec{a}\cdot\vec{b} = 8\times 7\times \cos 0 = 56$

e) $\vec{a}\cdot\vec{b} = 8\times 7 \cos \pi = -56$

f) $\vec{a}\cdot\vec{b} = 8\times 7 \cos 270 = 0$ (they're orthogonal)

$\boxed{\uparrow}$ **4** Show that the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\underset{\sim}{b} = 7\underset{\sim}{i} - 3\underset{\sim}{j}$ are perpendicular.

if $\vec{a} = x_1\vec{i} + y_1\vec{j}$ and $\vec{b} = x_2\vec{i} + y_2\vec{j}$ then $\vec{a}\cdot\vec{b} = x_1x_2 + y_1y_2$

So $\vec{a}\cdot\vec{b} = 3\times 7 + 7\times(-3) = 0$

So as $\vec{a}\cdot\vec{b} = 0$, it means $\vec{a}$ and $\vec{b}$ are perpendicular

5 Find the vector $\underset{\sim}{d}$ that is perpendicular to $\underset{\sim}{c} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$ and has a magnitude of 10.

$\vec{a} = x\vec{i} + y\vec{j}$ $\quad \vec{a}.\vec{c} = 0$ so we must have $4x - 3y = 0$

$\iff y = \frac{4x}{3}$

Further, $|\vec{a}| = 10$ so $\sqrt{x^2+y^2} = 10$

$\sqrt{x^2 + \frac{16x^2}{9}} = 10$ $\quad \sqrt{\frac{25x^2}{9}} = 10$ $\rightarrow$ or $\frac{5x}{3} = 10$ so $\begin{cases} x = 6 \\ y = 8 \end{cases}$ and

$\rightarrow$ or $-\frac{5x}{3} = 10$ so $\begin{cases} x = -6 \\ y = -8 \end{cases}$

so $\vec{a} = \pm(6\vec{i} + 8\vec{j})$

6 If the vectors $\underset{\sim}{e} = 7\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{f} = x\underset{\sim}{i} - 3\underset{\sim}{j}$ are perpendicular, find the value of x.

$\vec{e}.\vec{f} = 0$ so $7x + (-5)(-3) = 0$

so $x = \frac{-15}{7}$

7 If $\underset{\sim}{a} = -6\underset{\sim}{i} + 2\underset{\sim}{j}$, find: **(a)** $\underset{\sim}{a} \bullet \underset{\sim}{a}$ **(b)** $|\underset{\sim}{a}|$ **(c)** $\underset{\sim}{a} \bullet \underset{\sim}{a}$ in terms of $|\underset{\sim}{a}|$

a) $\vec{a} . \vec{a} = (-6)^2 + 2^2 = 40$

b) $|\vec{a}| = \sqrt{6^2 + 2^2} = \sqrt{40}$

c) $\vec{a} \bullet \vec{a} = [|\vec{a}|]^2$

8 For any vector $\underset{\sim}{a}$, find the value of each of the following, in terms of $|\underset{\sim}{a}|$ where necessary.
(a) $\underset{\sim}{a} \bullet \underset{\sim}{a}$ **(b)** $\hat{\underset{\sim}{a}} \bullet \hat{\underset{\sim}{a}}$ **(c)** $\underset{\sim}{a} \bullet (-\underset{\sim}{a})$

a) $\vec{a} = x\vec{i} + y\vec{j}$ $\quad \vec{a} . \vec{a} = x^2 + y^2 = [|\vec{a}|]^2$

b) $\hat{\underset{\sim}{a}} . \hat{\underset{\sim}{a}} = [|\hat{\underset{\sim}{a}}|]^2 = [1]^2 = 1$

c) $\vec{a} . (-\vec{a}) = -\vec{a} . \vec{a} = -[|\vec{a}|]^2$

10 Find the angle, correct to the nearest degree, between each of the following pairs of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$:
(a) $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} + 6\underset{\sim}{j}$ **(c)** $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$

a) $\vec{a} . \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ so $\cos\theta = \dfrac{\vec{a} . \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{9+10}{\sqrt{3^2+2^2} \times \sqrt{3^2+5^2}} = \dfrac{19}{\sqrt{13}\sqrt{34}} = \dfrac{19}{\sqrt{442}}$

$\cos\theta = \dfrac{19}{\sqrt{442}}$ $\quad \theta \simeq 25°$

b) $\cos\theta = \dfrac{\vec{a} . \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{-3 \times 5 + 2 \times 6}{\sqrt{3^2+2^2}\sqrt{5^2+6^2}} = \dfrac{-3}{\sqrt{13}\sqrt{61}} = \dfrac{-3}{\sqrt{793}}$ $\quad \theta \simeq 96°$

c) $\cos\theta = \dfrac{\vec{a} . \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{4 \times 3 - 1 \times 4}{\sqrt{4^2+1^2}\sqrt{3^2+4^2}} = \dfrac{8}{\sqrt{17}\sqrt{25}} = \dfrac{8}{5\sqrt{17}}$ $\quad \theta \simeq 67°$

11 Which vector is perpendicular to $\underset{\sim}{f} = -5\underset{\sim}{i} + 2\underset{\sim}{j}$ with magnitude 12?

A $\underset{\sim}{a} = \frac{12}{\sqrt{29}}\underbrace{(5\underset{\sim}{i} - 2\underset{\sim}{j})}_{=\vec{A}}$ B $\underset{\sim}{b} = \frac{12}{\sqrt{29}}\underbrace{(2\underset{\sim}{i} + 5\underset{\sim}{j})}_{\vec{B}}$ C $\underset{\sim}{c} = \frac{12}{\sqrt{29}}\underbrace{(2\underset{\sim}{i} - 5\underset{\sim}{j})}_{\vec{C}}$ D $\underset{\sim}{d} = \frac{12}{\sqrt{29}}\underbrace{(-2\underset{\sim}{i} + 5\underset{\sim}{j})}_{\vec{D}}$

$\underset{\sim}{f} \cdot \vec{A} = (-5) \times 5 + 2 \times (-2) \neq 0$ so not perpendicular

$\underset{\sim}{f} \cdot \vec{B} = -5 \times 2 + 2 \times 5 = 0$ so perpendicular

$\underset{\sim}{f} \cdot \vec{C} = -5 \times 2 - 5 \times 2 \neq 0$ not $\perp$

$\underset{\sim}{f} \cdot \vec{D} = -5 \times (-2) + 2 \times 5 \neq 0$ no $\perp$

so $\boxed{B}$

12 Vectors $\underset{\sim}{a} = x\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -6\underset{\sim}{i} + y\underset{\sim}{j}$ are perpendicular. What are possible values of x and y?

A $x = 1$ and $y = 3$ B $x = 1$ and $y = -3$ C $x = -2$ and $y = -6$ D $x = 2$ and $y = 6$

$x \times (-6) - 2 \times y = 0$ so $6x + 2y = 0$ or $3x + y = 0$

So $y = -3x$ they are of opposite signs

$\boxed{B}$

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14 The points A, B and C have position vectors $\overrightarrow{OA} = -2\underset{\sim}{i} - 3\underset{\sim}{j}$, $\overrightarrow{OB} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\overrightarrow{OC} = 8\underset{\sim}{i} - \underset{\sim}{j}$.

(a) Find the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ in component form.

(b) Find $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$ and $|\overrightarrow{AC}|$.

(c) Show that ΔABC is a right-angled triangle.

(d) Find the position vector of a point D such that $ABCD$ forms a square.

(e) Find the vector $\overrightarrow{BD}$, the other diagonal of the square $ABCD$.

(f) Show that the diagonals of the square $ABCD$ bisect at right angles.

a) $\vec{AB} = \vec{AO} + \vec{OB} = -\vec{OA} + \vec{OB} = -(-2\vec{i} - 3\vec{j}) + 2\vec{i} + 3\vec{j} = 4\vec{i} + 6\vec{j}$

$\vec{BC} = \vec{BO} + \vec{OC} = -\vec{OB} + \vec{OC} = -(2\vec{i} + 3\vec{j}) + (8\vec{i} - \vec{j}) = 6\vec{i} - 4\vec{j}$

$\vec{AC} = \vec{AO} + \vec{OC} = -\vec{OA} + \vec{OC} = -(-2\vec{i} - 3\vec{j}) + 8\vec{i} - \vec{j} = 10\vec{i} + 2\vec{j}$

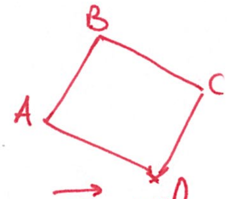
b) $|\vec{AB}| = \sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$

$|\vec{BC}| = \sqrt{6^2 + 4^2} = 2\sqrt{13}$

$|\vec{AC}| = \sqrt{10^2 + 2^2} = \sqrt{104} = 2\sqrt{26}$

d) $\vec{OD} = \vec{OA} + \vec{AD} = -2\vec{i} - 3\vec{j} + \vec{BC}$

$\vec{OD} = -2\vec{i} - 3\vec{j} + 6\vec{i} - 4\vec{j} = 4\vec{i} - 7\vec{j}$



c) for ΔABC to be a right-angled triangle, we must have $\vec{AB} \cdot \vec{BC} = 0$

$\vec{AB} \cdot \vec{BC} = 4 \times 6 + 6 \times (-4) = 0$ so indeed ΔABC right angled at B.

e) $\vec{BD} = \vec{BO} + \vec{OD} = -\vec{OB} + \vec{OD} = -(2\vec{i} + 3\vec{j}) + 4\vec{i} - 7\vec{j}$

So $\vec{BD} = 2\vec{i} - 10\vec{j}$

f) $\vec{BD} \cdot \vec{AC} = (2\vec{i} - 10\vec{j}) \cdot (10\vec{i} + 2\vec{j})$

$= 20 - 20$

$= 0$ so $\vec{BD}$ and $\vec{AC}$ are perpendicular