SOLVING TRIGONOMETRIC EQUATIONS

You will solve some simple trigonometric equations where the values are given in degrees. You will then solve equations where the answers are given in radians and learn how the question tells you the form of the answer.

To solve trigonometric equations you can use the same techniques that you have used for algebraic equations. You will also use trigonometric identities to help solve trigonometric equations.

Example 14

Solve each equation for $0° \le x \le 180°$.

(a) $3 + 2\sin x = 5\sin x$ **(b)** $\sin x = 2\cos x$ **(c)** $6\cos 2x = 3\sin 30°$

Solution

(a)

$$3 + 2\sin x = 5\sin x$$
$$3 = 3\sin x$$
$$\sin x = 1$$
$$x = 90°$$

(b)

$$\sin x = 2\cos x$$
$$\frac{\sin x}{\cos x} = 2$$
$$\tan x = 2$$
$$x = 63.43°$$
$$x = 63°26'$$

(c)

$$6\cos 2x = 3\sin 30°$$
$$\cos 2x = \frac{3\sin 30°}{6}$$
$$\cos 2x = 0.25$$
$$2x = 75.52°,\ 360° - 75.52°$$
$$2x = 75.52°,\ 284.48°$$
$$x = 37.76°,\ 142.24°$$
$$x = 37°46',\ 142°14'$$

Looking at the solution to **(c)** above, you might ask, 'Why two answers?' Consider that the value of x is given as between 0° and 180°, so the value of $2x$ is between 0° and $2 \times 180° = 360°$. There are thus four quadrants in which the cosine function might have solutions, but you also know that the solutions must be positive. This means that the answers can be in the first and the fourth quadrants.

Example 15

Solve each equation for $0° \le x \le 360°$.

(a) $\sec^2 x = 2$ **(b)** $2\sin x - \tan x = 0$ **(c)** $\sin x - \cos x = 0$

Solution

(a)

$$\sec^2 x = 2$$
$$\sec x = \pm\sqrt{2}$$

This gives values of x in all 4 quadrants.

In the first quadrant, $\sec x = \sqrt{2}$ gives $x = 45°$.

$$\therefore\ x = 45°,\ 180° - 45°,\ 180° + 45°,\ 360° - 45°$$
$$x = 45°,\ 135°,\ 225°,\ 315°$$

(b)

$$2\sin x - \tan x = 0$$
$$2\sin x - \frac{\sin x}{\cos x} = 0$$
$$\sin x\left(\frac{2\cos x - 1}{\cos x}\right) = 0$$
$$\sin x(2\cos x - 1) = 0,\ \cos x \ne 0$$
$$\sin x = 0,\ \cos x = 0.5$$
$$x = 0°,\ 180°,\ 360° \quad \text{or} \quad x = 60°,\ 300°$$

(c)

$$\sin x - \cos x = 0$$
$$\sin x = \cos x$$
$$\tan x = 1$$
$$x = 45°,\ 180° + 45°$$
$$x = 45°,\ 225°$$

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The following equations give their domain in terms of π, so the answers are to be given in radians.

You will need to remember the exact values for the common trigonometric ratios, or how to calculate them. You also need to know the signs of the values in the four quadrants (remember 'ASTC').

Example 16

Solve each equation for $0 \leq x \leq 2\pi$.

(a) $\cos x = \frac{\sqrt{3}}{2}$ **(b)** $\sin^2 x + 2\sin x + \cos^2 x = 0$ **(c)** $\sec x = -2$

Solution

(a)

Solve: $\cos x = \frac{\sqrt{3}}{2}$

From the sign, angles are in 1st and 4th quadrants: $x = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$x = \frac{\pi}{6}, \frac{11\pi}{6}$

(b)

Solve: $\sin^2 x + 2\sin x + \cos^2 x = 0$

Rearrange equation: $\sin^2 x + \cos^2 x + 2\sin x = 0$

Use the identity $\sin^2 x + \cos^2 x = 1$: $1 + 2\sin x = 0$

$2\sin x = -1$

$\sin x = -\frac{1}{2}$

From the sign, angles are in 3rd and 4th quadrants: $x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

(c)

Solve: $\sec x = -2$

Change to $\cos x$: $\frac{1}{\cos x} = -2$

$\cos x = -\frac{1}{2}$

From the sign, angles are in 2nd and 3rd quadrants: $x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Example 18

Solve the equation $\sin\theta = \frac{4\sin\frac{\pi}{9}}{5}$ if θ is an angle in a triangle. Give your answer correct to 2 decimal places.

Solution

Evaluate the expression to find θ: $\theta = 0.277$

Because θ is an angle in a triangle, consider also: $\theta = \pi - 0.277 = 2.868$

The original equation can be rearranged into $\frac{4}{\sin\theta} = \frac{5}{\sin\frac{\pi}{9}}$, which is a statement of the sine rule (see Section 2.8 The sine rule, page 47). This implies that $\frac{\pi}{9}$ is also an angle of the triangle so the second value is too large (because $2.868 + \frac{\pi}{9} > \pi$ will not fit), hence: $\theta = 0.28$ (to 2 d.p.)

Example 17

Solve, for $-\pi \le \theta \le \pi$: **(a)** $2\sin^2\theta = 1$ **(b)** $3\sin\theta = 2\cos\theta$ **(c)** $\sin^2\theta + 2\sin\theta + 1 = 0$

Solution

(a)

Solve: $2\sin^2\theta = 1$

Rearrange: $\sin^2\theta = \frac{1}{2}$

Square root of each side: $\sin\theta = \pm\frac{1}{\sqrt{2}}$

From the sign, angles are in all quadrants: $\theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ as $-\pi \le \theta \le \pi$

(b)

Solve: $3\sin\theta = 2\cos\theta$

Divide both sides by $3\cos\theta$: $\frac{\sin\theta}{\cos\theta} = \frac{2}{3}$

But $\tan\theta = \frac{\sin\theta}{\cos\theta}$: $\tan\theta = \frac{2}{3}$

From the sign, angles are in 1st and 3rd quadrants: $\theta = 0.5880, -\pi + 0.5880$ as $-\pi \le \theta \le \pi$

$\theta = -2.5536, 0.5880$ (to 4 d.p.)

(c)

Solve: $\sin^2\theta + 2\sin\theta + 1 = 0$

Factorise: $(\sin\theta + 1)^2 = 0$

Square root of each side: $\sin\theta + 1 = 0$

Rearrange: $\sin\theta = -1$

From the sign, angles are in 3rd and 4th quadrants: $\theta = -\frac{\pi}{2}$ as $-\pi \le \theta \le \pi$

Example 19

Solve, for $0 \le x \le \pi$: **(a)** $\cos\frac{x}{2} = \frac{1}{\sqrt{2}}$ **(b)** $\sin 2x = -\frac{1}{2}$ **(c)** $\sec^2 2x - 2\tan 2x = 0$

Solution

(a)

Solve: $\cos\frac{x}{2} = \frac{1}{\sqrt{2}}$

$0 \le x \le \pi$ so $0 \le \frac{x}{2} \le \frac{\pi}{2}$: $\frac{x}{2} = \frac{\pi}{4}$

$x = \frac{\pi}{2}$

(b)

Solve: $\sin 2x = -\frac{1}{2}$

$0 \le x \le \pi$ so $0 \le 2x \le 2\pi$: $2x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$2x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$x = \frac{7\pi}{12}, \frac{11\pi}{12}$

(c)

Solve: $\sec^2 2x - 2\tan 2x = 0$

Use identity $\sec^2 2x = 1 + \tan^2 2x$: $1 + \tan^2 2x - 2\tan 2x = 0$

Rearrange: $\tan^2 2x - 2\tan 2x + 1 = 0$

Factorise: $(\tan 2x - 1)^2 = 0$

$\tan 2x = 1$

$0 \le x \le \pi$ so $0 \le 2x \le 2\pi$: $2x = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$x = \frac{\pi}{8}, \frac{5\pi}{8}$

Example 20

Solve for $-\pi \le \theta \le \pi$: **(a)** $\sec^2 2\theta - 2\tan 2\theta = 0$ **(b)** $\sin^2\theta + 2\sin\theta + \cos^2\theta = 0$

Solution

(a)

Solve: $\sec^2 2\theta - 2\tan 2\theta = 0$

Use $\sec^2 x = 1 + \tan^2 x$: $1 + \tan^2 2\theta - 2\tan 2\theta = 0$

Rearrange: $\tan^2 2\theta - 2\tan 2\theta + 1 = 0$

Factorise: $(\tan 2\theta - 1)^2 = 0$

$$\tan 2\theta - 1 = 0$$

$$\tan 2\theta = 1$$

$$2\theta = -2\pi + \frac{\pi}{4}, -\pi + \frac{\pi}{4}, \frac{\pi}{4}, \pi + \frac{\pi}{4}, \text{ as } -2\pi \le 2\theta \le 2\pi.$$

$$2\theta = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\theta = -\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$$

(b) $\sin^2\theta + 2\sin\theta + \cos^2\theta = 0$

Use $\sin^2\theta + \cos^2\theta = 1$: $1 + 2\sin\theta = 0$

$$2\sin\theta = -1$$

$$\sin\theta = -\frac{1}{2}$$

$$\theta = -\pi + \frac{\pi}{6}, -\frac{\pi}{6}$$

$$\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}$$

Example 21

Solve the equation $\operatorname{cosec}^2 x - 2\cot x = 4$, for $0 \le x \le 2\pi$.

Solution

Use $\operatorname{cosec}^2 x = 1 + \cot^2 x$: $1 + \cot^2 x - 2\cot x = 4$

Rearrange: $\cot^2 x - 2\cot x - 3 = 0$

Factorise: $(\cot x - 3)(\cot x + 1) = 0$

$\cot x = 3$ is the same as $\tan x = \frac{1}{3}$: $x = 0.32, \pi + 0.32$

$$x = 0.32, 3.46$$

$\cot x = -1$ is the same as $\tan x = -1$: $x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

The solutions in order are: $x = 0.32, \frac{3\pi}{4}, 3.46, \frac{7\pi}{4}$