SERIES AND SIGMA NOTATION (Σ)

Series

When you write a set of numbers in order according to some rule, the addition of all numbers is called a **series** and each number in the set is called a **term** of the series. In other words, a series is the sum of the terms of its corresponding sequence.

For example, $1 + 4 + 9 + 16 + \ldots + n^2$ is a series in which the first term is 1, the second term is 4, the third term is 9 and the nth term is n^2.

The symbols T_n or u_n are commonly used to represent the nth term of a series. Thus in the series $1 + 4 + 9 + 16 + \ldots + n^2$, $T_1 = 1$, $T_2 = 4$, $T_3 = 9$ and $T_n = n^2$. T_n is also called the **general term** of the series, because it allows you to find any other term of the series according to its rule.

$S_n = T_1 + T_2 + T_3 + \ldots + T_n$ is the sum of the series.

A useful result is that $S_{n-1} = T_1 + T_2 + T_3 + \ldots + T_{n-1}$

so that $S_n = S_{n-1} + T_n$

and hence $T_n = S_n - S_{n-1}$

Example 12

What are the first three terms of the series whose general term is given by $T_n = n^2 + 2$?

Solution

$T_1 = 1^2 + 2 = 3 \qquad T_2 = 2^2 + 2 = 6 \qquad T_3 = 3^2 + 2 = 11$

This series can be written as $S_n = 3 + 6 + 11 + \ldots + (n^2 + 2)$.

Example 13

Find the nth term of the series $4 + 7 + 10 + 13 + \ldots$

Solution

Because the terms increase by 3 each time, you can rewrite each term as a number plus an increasing multiple of 3:

$T_1 = 4$	$T_2 = 7$	$T_3 = 10$
$= 1 + 3$	$= 1 + 6$	$= 1 + 9$
$= 1 + 3 \times 1$	$= 1 + 3 \times 2$	$= 1 + 3 \times 3$

$\therefore\ T_n = 1 + 3n = 3n + 1$

Sigma notation (Σ)

The symbol Σ (the Greek capital letter sigma) is used in mathematics to mean 'the sum of'. When you sum the terms of a sequence you have a series, so sigma notation provides a shorter way to write the series.

You can rewrite the series discussed above using sigma notation:

$$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + 16 + \ldots + n^2$$

$$\sum_{k=1}^{n} \left(k^2 + 2\right) = 3 + 6 + 11 + \ldots + (n^2 + 2)$$

$$\sum_{k=1}^{n} (3k + 1) = 4 + 7 + 10 + \ldots + (3n + 1)$$

SERIES AND SIGMA NOTATION (Σ)

Each sigma means 'the sum of these terms, starting with the value underneath the Σ and continuing to the value above the Σ'. For example:

- $\sum_{n=1}^{5} x^n$ denotes the sum of terms of the form x^n, where n has the values 1, 2, 3, 4, 5

  Thus: $\sum_{n=1}^{5} x^n = x^1 + x^2 + x^3 + x^4 + x^5$

- $\sum_{k=0}^{n} (2k+1)$ denotes the sum of terms of the form $2k + 1$, where $k = 0, 1, 2, 3, \ldots n$

  (so k is an integer from 0 to n inclusive): $\sum_{k=0}^{n} (2k+1) = 1 + 3 + 5 + \ldots + (2n+1)$

  The right-hand side is the expansion of the series defined by $\sum_{k=0}^{n} (2k+1)$.

- $\sum_{r=1}^{10} rx^{r-1} = 1 + 2x + 3x^2 + \ldots + 10x^9$

It is also useful to use sigma notation to obtain a shorter expression for the sum of a number of terms.

- $1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{k=1}^{n} k^2$ The left-hand expression has n terms.
- $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + 10 \times 11 = \sum_{n=1}^{10} n(n+1)$ The left-hand expression has 10 terms.
- $2 + 2x + 2x^2 + \ldots + 2x^8 = \sum_{k=0}^{8} 2x^k$ The left-hand expression has 9 terms.

Example 14

Evaluate: **(a)** $\sum_{n=2}^{5} n^2$ **(b)** $\sum_{n=1}^{5} (2n-1)$

Solution

Write the series in full, then add the terms.

(a)
$$\sum_{n=2}^{5} n^2 = 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54$$

(b)
$$\sum_{n=1}^{5} (2n-1) = 1 + 3 + 5 + 7 + 9 = 25$$