THE SECOND DERIVATIVE AND TURNING POINTS

Just as you compared the graphs of $f(x)$ and $f'(x)$ drawn together on the same horizontal scale, you now need to consider the graphs of $f(x)$, $f'(x)$ and $f''(x)$ together.

Consider: $y = f(x) = x^3 - x^2 - x + 1$

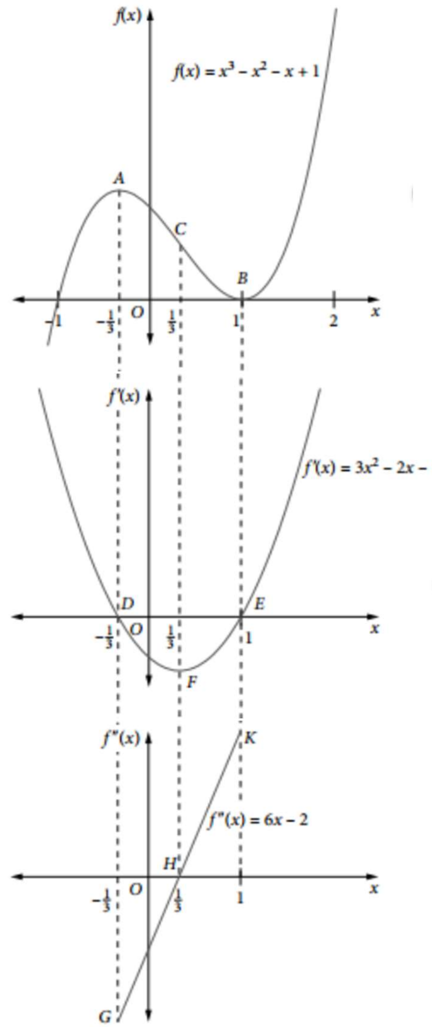
Differentiate: $\dfrac{dy}{dx} = y' = f'(x) = 3x^2 - 2x - 1$

Differentiate again: $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} = y'' = f''(x) = 6x - 2$

For stationary points, $f'(x) = 0$: $3x^2 - 2x - 1 = 0$

Factorise: $(3x+1)(x-1) = 0$

Solve: $x = -\frac{1}{3}, 1$



On the graph, the vertical lines ADG and BEK correspond to the lines $x = -\frac{1}{3}$ and $x = 1$ respectively.

Using the first derivative test (or the graph of $f'(x)$), you can say that A is a maximum turning point and that B is a minimum turning point.

- Consider where the line ADG cuts the three curves. The point A on $f(x)$ is a maximum turning point. The point D on $f'(x)$ is where $f'(x)$ cuts the x-axis, that is, $f'(x) = 0$. At the point G on $f''(x)$, $f''(x) < 0$. Hence the curve $y = f(x)$ is concave down at a maximum turning point.
- Consider where the line BEK cuts the three curves: at point B, $f(x)$ is a minimum turning point; at point E, $f'(x)$ cuts the x-axis, i.e. $f'(x) = 0$; at point K, $f''(x) > 0$. Hence the curve $y = f(x)$ is concave up at a minimum turning point.
- Consider where the line CFH cuts the three curves: at point C, $f(x)$ seems to have its steepest tangent; at point F, $f'(x)$ has its least value, i.e. the gradient of $f(x)$ is at its most negative (steepest); at point H, $f''(x) = 0$. The concavity changes either side of C, so C is a point of inflection on $y = f(x)$.

For $y = f(x) = x^3 - x^2 - x + 1$, you can say that the function has a maximum turning point at $\left(-\frac{1}{3}, 1\frac{5}{27}\right)$, a minimum turning point at $(1, 0)$, is concave down for $x < \frac{1}{3}$, is concave up for $x > \frac{1}{3}$ and has a point of inflection at $\left(\frac{1}{3}, \frac{16}{27}\right)$.

Second derivative test for turning points

- If $\dfrac{dy}{dx} = 0$ at (x_1, y_1) and $\dfrac{d^2y}{dx^2} < 0$ then the point (x_1, y_1) is a maximum turning point.
- If $\dfrac{dy}{dx} = 0$ at (x_2, y_2) and $\dfrac{d^2y}{dx^2} > 0$ then the point (x_2, y_2) is a minimum turning point.
- If $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} = 0$ at (x_3, y_3) then the point may be a turning point **OR** it may be a horizontal point of inflection. Further investigation is needed.

Point of inflection test

- If $\frac{d^2y}{dx^2}=0$ at (x_4, y_4) and the concavity changes either side of this point, then (x_4, y_4) is a point of inflection.
- If $\frac{dy}{dx}=0$ at a point of inflection then it is called a horizontal point of inflection.

Example 9

Investigate the stationary points of:

(a) $y=x^4$ **(b)** $y=3x^2-3x-x^3$

Solution

(a) $y=x^4$: $\frac{dy}{dx}=4x^3$; $\frac{d^2y}{dx^2}=12x^2$

For stationary points, $\frac{dy}{dx}=0$: $x=0$

At $x=0$, $\frac{d^2y}{dx^2}=0$, so you can't identify the nature of this stationary point.

Check the sign of the first derivative: For $x<0$, e.g. $x=-1$: $\frac{dy}{dx}=-4<0$

For $x>0$, e.g. $x=1$: $\frac{dy}{dx}=4>0$

The gradient changes from negative to positive through $x=0$, so $(0,0)$ is a minimum turning point.

Alternative method:

Check the sign of the second derivative: For $x<0$, e.g. $x=-1$: $\frac{d^2y}{dx^2}=12>0$

For $x>0$, e.g. $x=1$: $\frac{d^2y}{dx^2}=12>0$

The curve is concave up on both sides of the stationary point, so $(0,0)$ is a minimum turning point.

(b) $y=3x^2-3x-x^3$: $\frac{dy}{dx}=6x-3-3x^2$ $\frac{d^2y}{dx^2}=6-6x$

$$=-3\left(x^2-2x+1\right)$$

$$=-3(x-1)^2$$

For stationary points, $\frac{dy}{dx}=0$: $(x-1)^2=0$, hence $x=1$

At $x=1$, $\frac{d^2y}{dx^2}=0$, so you can't identify the nature of this stationary point.

Check the sign of the second derivative: For $x<1$, e.g. $x=0.1$: $\frac{d^2y}{dx^2}=6-0.6$

$$=5.4>0$$

For $x>1$, e.g. $x=1.1$: $\frac{d^2y}{dx^2}=6-6.6$

$$=-0.6<0$$

The concavity changes either side of $x=1$, so $(1,-1)$ is a horizontal point of inflection.

Example 10

Consider the graph of $y = 2x^4 - x + 1$.

(a) Find any turning points and points of inflection.

(b) For what values of x is the curve concave up?

(c) On the same set of axes sketch y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution

(a) $y = 2x^4 - x + 1$: $\quad \frac{dy}{dx} = 8x^3 - 1$

For stationary points, $\frac{dy}{dx} = 0$: $\quad 8x^3 - 1 = 0$, hence $x^3 = \frac{1}{8}$ and so $x = \frac{1}{2}$.

At $x = \frac{1}{2}$, $y = 2 \times \frac{1}{2^4} - \frac{1}{2} + 1$

$= 1\frac{3}{8}$ $\quad \therefore$ stationary point is at $\left(\frac{1}{2}, 1\frac{3}{8}\right)$.

Find $\frac{d^2y}{dx^2}$: $\quad \frac{d^2y}{dx^2} = 24x^2$

At $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 6 > 0 \quad \therefore \left(\frac{1}{2}, 1\frac{3}{8}\right)$ is a minimum turning point.

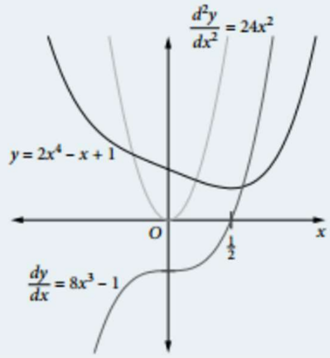
$\frac{d^2y}{dx^2} = 0$ at $x = 0$, so check concavity to see if there is an inflection point at $(0, 1)$:

$x < 0$: $\frac{d^2y}{dx^2} > 0 \qquad x > 0$: $\frac{d^2y}{dx^2} > 0$

The concavity does not change either side of $x = 0$, so the curve does not have a point of inflection at $(0, 1)$.

(b) Concave up when $\frac{d^2y}{dx^2} > 0$, hence concave up for all real x.

(c)



This example shows that $\frac{d^2y}{dx^2} = 0$ is not enough to confirm a point of inflection. You must also check that the concavity changes at the point.

Example 11

Sketch the graph of $y = x^3(4 - x)$, showing any turning points and points of inflection. For what values of x is the curve concave down?

Solution

$y = 4x^3 - x^4$: $\quad \frac{dy}{dx} = 12x^2 - 4x^3 = 4x^2(3 - x) \quad \frac{d^2y}{dx^2} = 24x - 12x^2 = 12x(2 - x)$

For stationary points, $\frac{dy}{dx} = 0$: $\quad 4x^2(3 - x) = 0$

$x = 0$ or 3

$y = 0$ or 27 $\quad \therefore$ stationary points at $(0, 0)$ and $(3, 27)$

At $x = 0$, $\frac{d^2y}{dx^2} = 0$. Investigate further: $\quad x = -1$: $\frac{d^2y}{dx^2} = -12(2 + 1) < 0$

$x = 1$: $\frac{d^2y}{dx^2} = 12(2 - 1) > 0$

The sign of $\frac{d^2y}{dx^2}$ changes, so the concavity changes. Hence $(0, 0)$ is a horizontal point of inflection.

At $x = 3$: $\quad \frac{d^2y}{dx^2} = 36(2 - 3) < 0$ $\quad$ Hence $(3, 27)$ is a relative maximum turning point.

For points of inflection, $\frac{d^2y}{dx^2} = 0$: $\quad 12x(2 - x) = 0$

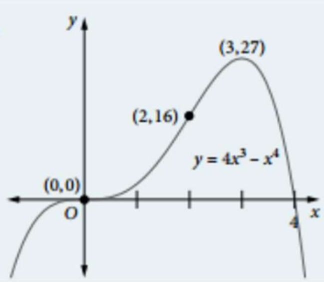
$x = 0$ or 2

$(0, 0)$ has already been identified as a horizontal point of inflection. For $(2, 16)$:

$x = 1$: $\frac{d^2y}{dx^2} = 12(2 - 1) > 0 \qquad x = 3$: $\frac{d^2y}{dx^2} = 36(2 - 3) < 0$

The sign of $\frac{d^2y}{dx^2}$ changes, so the concavity changes. Hence $(2, 16)$ is a point of inflection.

Curve is concave downwards for $x < 0$ and $x > 2$.



Summary of tests for turning points and points of inflection

Stationary point: $\frac{dy}{dx} = 0$

Using the first derivative only:

Turning point: $\frac{dy}{dx} = 0$, $\frac{dy}{dx}$ changes sign on passing through the stationary point.

Maximum turning point: $\frac{dy}{dx} = 0$, $\frac{dy}{dx}$ changes sign from positive to negative on passing through the stationary point.

Minimum turning point: $\frac{dy}{dx} = 0$, $\frac{dy}{dx}$ changes sign from negative to positive on passing through the stationary point.

Using the first and second derivatives:

Turning point: $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2}$ does not change sign on passing through the stationary point.

Maximum turning point: $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} < 0$ at the stationary point.

Minimum turning point: $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} > 0$ at the stationary point.

Point of inflection: $\frac{d^2y}{dx^2} = 0$, $\frac{d^2y}{dx^2}$ changes sign on passing through the stationary point (i.e. concavity changes).

Special situation:

$\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$ at a point: may be a turning point **or** a horizontal point of inflection. You must check to see if either the gradient or the concavity changes.

Global maxima and minima

In Example **10** the minimum turning point gives the least value of the function over its domain. It is called the global minimum value of the function. The function has no greatest value.

In Example **11** the maximum turning point gives the greatest value of the function over its domain. It is called the global maximum value of the function. The function has no least value.

The greatest and least values of a function, also called the global maxima and global minima, may occur at the endpoints of the domain or at the turning points of the function.

When asked to find the global maximum or global minimum value of a function, as well as considering the values of the function at the turning points, you also need to find the value of the function at the endpoints of the given domain (or the natural domain if no restrictions are given). You then compare the value of the function at these points to find the global maximum and the global minimum.

Example 12

(a) Find the coordinates of the stationary points of $y = x^2e^x$ and determine their nature.

(b) Find the coordinates of any points of inflection.

(c) Sketch the graph of this function.

(b) Find the global maximum and global minimum values of this function.

Solution

(a) $y = x^2e^x$

Differentiate: $\frac{dy}{dx} = 2xe^x + x^2e^x$

Factorise: $\frac{dy}{dx} = xe^x(2 + x)$

For stationary points, $\frac{dy}{dx} = 0$: $x = 0, -2$

$y = 0, 4e^{-2}$

Find second derivative: $\frac{d^2y}{dx^2} = 2e^x + 2xe^x + 2xe^x + x^2e^x$

Factorise: $\frac{d^2y}{dx^2} = e^x\left(2 + 4x + x^2\right)$

$x = 0$: $\frac{d^2y}{dx^2} = 2 > 0$ so minimum turning point at (0, 0).

$x = -2$: $\frac{d^2y}{dx^2} = -2e^{-2} < 0$ so maximum turning point at $(0, 4e^{-2})$.

(b) Require $\frac{d^2y}{dx^2} = 0$: $e^x\left(2 + 4x + x^2\right) = 0$

$e^x > 0$: $x^2 + 4x + 2 = 0$

$$x = \frac{-4 \pm \sqrt{8}}{2}$$

$$= -2 \pm \sqrt{2}$$

$$\approx -0.586, -3.41$$

$x = -1$: $\frac{d^2y}{dx^2} = -e^{-1} < 0$

$x = 0$: $\frac{d^2y}{dx^2} = 2 > 0$ and concavity changes at $x = -0.586$

Hence a point of inflection when $x = -0.586$

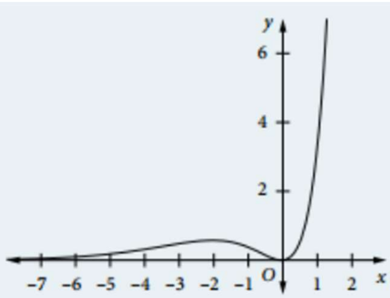
$x = -4$: $\frac{d^2y}{dx^2} = 2e^{-4} > 0$

$x = -3$: $\frac{d^2y}{dx^2} = -e^{-3} < 0$ and concavity changes at $x = -3.41$

Hence a point of inflection when $x = -3.41$

The points of inflection are (−0.586, 0.191) and (−3.41, 0.384).

(c)



(d) From the graph the global minimum value is 0. There is no global maximum value.

Algebraically, as $x \to -\infty$, $y \to 0$ from above (asymptote).

As $x \to \infty$, y increases without bound. Hence the global minimum value occurs at the minimum turning point.