PERMUTATIONS

Simplify

${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$	${}^4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$	${}^9P_1 = \frac{9!}{(9-1)!} = \frac{9!}{8!} = 9$	${}^{10}P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!} = 10!$	${}^nP_4 = \frac{n!}{(n-4)!}$
$5! = 120$	$\frac{7!}{6!} = 7$	$\frac{10!}{8!} = 10 \times 9 = 90$	$\frac{n!}{(n-2)!} = n(n-1)$	

3 How many different arrangements can be made using three of the letters of the word SUNDAY?

6 letters, no repetition, so it's ${}^6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 120$

4 In how many different ways can five people be arranged in a row?

A 20 **B** 60 $\boxed{\textbf{C}}$ 120 **D** 720

$5! = 120$

6 How many different arrangements of the letters of the word MINOR are possible if:

(a) the two vowels are next to each other **(b)** the first and last letters are consonants?

(a) if we put a vowel in the 1st place, then we know that the other vowel goes 2nd, then 3! for the consonants. So $2 \times 3!$ Then we repeat that for the 2nd place, 3rd place, 4th place, so $4 \times 2 \times 3! = 48$

(b) 1st position: 3 possibilities, then 2 possibilities for last position (6 in total). Then 3! to arrange the remaining letters in the positions 2, 3, 4. So $6 \times 3! = 36$

10 Five different magazines and four different books are arranged in a row with the books standing together. Indicate whether each statement below is a correct or incorrect step in the calculation of the total number of arrangements.

(a) $5! \times 5!$ $\boxed{\textbf{(b)}}$ $6! \times 4!$ **(c)** $9!$ $\boxed{\textbf{(d)}}$ 17 280

4! arrangements for the 4 books. Then the 1st book can be in position 1 to 6. Then 5! possible arrangements for magazines, so $6 \times 4! \times 5! = 17280$ [so $6 \times 4!$

13 How many numbers greater than 4000 can be formed using the digits 3, 5, 7, 8, 9 if repetition is not allowed?

The numbers can be made of 4 or 5 digits.

if 4 digits: 4 choices for the 1st digit (not 3) then $4 \times 3 \times 2$ so $4 \times 24 = 96$

if 5 digits: 5! possibilities

So in total: $96 + 5! = 216$.

15 In how many ways can five different Mathematics books, four different Physics books and two different Chemistry books be arranged on a shelf if the books in each subject must be together?

3 choices of subject for the 1st group, then 2, then 1. So $3!$ possible arrangements of subjects.
Then, for each subject, $5!$ possible arrangement for Maths, $4!$ for Physics, and $2!$ for Chemistry.
So in total, $3! \times 4! \times 5! \times 2! = 34560$ possibilities.

16 In how many ways can three doctors, three nurses and three patients be arranged in a row if the three patients must be together?

The 1st of the 3 patients can be in position 1 to 7. Then $3!$ arrangements. So $7 \times 3!$ for the patients.
Then, $6!$ possible arrangements for the doctors and nurses.

So in total, $7 \times 3! \times 6! = 30,240$ possibilities

19 A car holds three people in the front seat and four in the back seat. In how many ways can seven people be seated in the car if two particular people must sit in the back seat and one particular person is the driver?

The driver is seated in the driver seat, so only 1 possibility.
Then, for the 2 people that must seat at the back, $4 \times 3 = 12$ possibilities
Then, for the last 4, $4 \times 3 \times 2 \times 1 = 4!$ possibilities

So in total, $12 \times 4! = 288$ possibilities.

20 If ${}^6P_r = 120$, find the value of r.

$\dfrac{6!}{(6-r)!} = 120$, so $(6-r)! = \dfrac{6!}{120} = \dfrac{720}{120} = 6 = 3!$

so we must have $6 - r = 3$ so $r = 3$

23 A father, a mother and six children stand in a ring. In how many ways can they be arranged if the father and the mother do not stand together?

Once the mum has been placed, there are 5 possibilities for the dad.

Then $6!$ for the children.

So in total, $5 \times 6! = 3,600$ possibilities.

25 The ratio of the number of arrangements of $(2n + 2)$ different objects taken n at a time to the number of arrangements of $2n$ different objects taken n at a time is $14:5$. Find the value of n.

$$\frac{^{2n+2}P_n}{^{2n}P_n} = \frac{14}{5} \quad \text{so} \quad 5 \times {}^{2n+2}P_n = 14 \times {}^{2n}P_n$$

$$\Leftrightarrow 5 \times \frac{(2n+2)!}{[(2n+2)-n]!} = 14 \times \frac{(2n)!}{[2n-n]!}$$

$$\Leftrightarrow 5 \times \frac{(2n+2)(2n+1) \times \cancel{(2n)!}}{(n+2)(n+1) \times \cancel{n!}} = \frac{14\ \cancel{(2n)!}}{\cancel{n!}}$$

$$\Leftrightarrow \frac{5 \times 2\cancel{(n+1)}(2n+1)}{(n+2)\cancel{(n+1)}} = 14 \quad \Leftrightarrow 10(2n+1) = 14(n+2)$$

$$\Leftrightarrow (20-14)n = 28-10 = 18$$

$$\Leftrightarrow 6n = 18 \quad \text{so} \quad \boxed{n=3}$$

26 If $^{2n}P_n = 8 \times {}^{2n-1}P_{n-1}$, find the value of n.

$$\Leftrightarrow \frac{(2n)!}{n!} = \frac{8\ (2n-1)!}{[(2n-1)-(n-1)]!}$$

$$\Leftrightarrow \frac{2n}{n!} = \frac{8}{n!} \quad \Leftrightarrow \quad 2n = 8 \quad \text{so} \quad \boxed{n=4}$$

27 Prove from the formula for ^nP_r that: $^{n+1}P_r = {}^nP_r + r \times {}^nP_{r-1}$

We start from RHS as it's more complex.

$$RHS = \frac{n!}{(n-r)!} + r \times \frac{n!}{[n-(r-1)]!}$$

$$RHS = \frac{n!}{(n-r)!} + r \times \frac{n!}{(n-r+1)!} = \frac{n! \times (n-r+1)}{(n-r+1)!} + r \times \frac{n!}{(n-r+1)!}$$

$$RHS = \frac{(n+1)n! - rn! + rn!}{(n-r+1)!} = \frac{(n+1)!}{(n+1-r)!}$$

$$\text{So } RHS = \frac{(n+1)!}{[(n+1)-r]!} = {}^{n+1}P_r = LHS. \quad \text{proven}$$

28 Show that: ${}^{n}P_{r} = {}^{n-2}P_{r} + 2r \times {}^{n-2}P_{r-1} + r(r-1) \times {}^{n-2}P_{r-2}$ we start from the RHS which is the most complex

$$RHS = \frac{(n-2)!}{[(n-2)-r]!} + 2r \times \frac{(n-2)!}{[(n-2)-(r-1)]!} + r(r-1) \times \frac{(n-2)!}{[(n-2)-(r-2)]!}$$

$$RHS = \frac{(n-2)!}{(n-r-2)!} + 2r \frac{(n-2)!}{(n-r-1)!} + r(r-1) \times \frac{(n-2)!}{(n-r)!}$$

$$RHS = (n-2)! \left[\frac{(n-r)(n-r-1) + 2r(n-r) + r(r-1)}{(n-r)!} \right]$$

$$RHS = \frac{(n-2)!}{(n-r)!} \left[n^2 - \cancel{rn} - n - \cancel{rn} + \cancel{r^2} + \cancel{r} + 2\cancel{rn} - 2\cancel{r^2} + \cancel{r^2} - \cancel{r} \right]$$

$$RHS = \frac{(n-2)!}{(n-r)!} \times \left[n^2 - n \right]$$

$$RHS = \frac{(n-2)!}{(n-r)!} \times n \times (n-1) = \frac{n!}{(n-r)!} = {}^{n}P_{r} = LHS$$

proven

29 In how many ways can five writers and five artists be arranged in a circle so that the writers are separated? In how many ways can this be done if two particular artists must not sit next to a particular writer?

Once a writer is seated, there are 4! choices for seating the other 4 writers.
Then for the artists, there are 5! choices.
So 4! × 5! = 2880.

For the 2nd question, same for the writers: 4!.
Then, for the artists, first we seat the 2 artists that cannot seat next to a particular writer: 3 × 2 = 6 possibilities. Then the remaining ones, which is 3! possibilities.
So 4! × 6 × 3! = 864

HSC 2018

8 Six men and six women are to be seated at a round table.

In how many different ways can they be seated if men and women alternate?

A. 5! 5!

B. 5! 6!

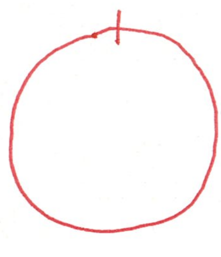
C. 2! 5! 5!

D. 2! 5! 6!

5! for men

6! for women

so $\boxed{B}$



PERMUTATIONS

4. Combinatorics, EXT1 A1 2012 HSC 5 MC

7 letters.

How many arrangements of the letters of the word $OLYMPIC$ are possible if the C and the L are to be together in any order?

(A) $5!$

(B) $6!$

(C) $2 \times 5!$

$\boxed{\text{(D)}}$ $2 \times 6!$

Since C and L must be kept together, there are 6 possibilities for the 1st of the 2, and we need to multiply by 2 as it could be CL or LC.

So 6×2. Then, $5!$ for the remaining letters.

So in total, $6 \times 2 \times 5! = 2 \times 6!$ $\boxed{D}$

30. Combinatorics, EXT1 A1 2006 HSC 3c

Sophie has five coloured blocks: one red, one blue, one green, one yellow and one white. She stacks two, three, four or five blocks on top of one another to form a vertical tower.

i. How many different towers are there that she could form that are three blocks high? ***(1 mark)***

ii. How many different towers can she form in total? ***(2 marks)***

i) 5 possibilities for the 1st block, then 4 possibilities for the 2nd block, then 3 possibilities for the 3rd block.

So $5 \times 4 \times 3 = 60$ in total.

ii) Height 2: 5 possibilities for 1st colour, then 4 for 2nd colour so $5 \times 4 = 20$

Height 3: 5 possibilities for 1st colour, 4 for 2nd, 3 for 3rd so $5 \times 4 \times 3 = 60$

Height 4: 5 possibilities for 1st colour, 4 for 2nd, 3 for 3rd, 2 for 4th so $5 \times 4 \times 3 \times 2 = 120$

Height 5: $5 \times 4 \times 3 \times 2 \times 1 = 120$

TOTAL $= 20 + 60 + 120 + 120 = 320$ possibilities.

PERMUTATIONS

HSC EXT 2017 (not permutations)

10 Three squares are chosen at random from the 3×3 grid below, and a cross is placed in each chosen square.

1	2	
	3	

What is the probability that all three crosses lie in the same row, column or diagonal?

A. $\frac{1}{28}$

$\boxed{\text{B.}}$ $\frac{2}{21}$

C. $\frac{1}{3}$

D. $\frac{8}{9}$

4 times: if 1st choice is 1 (Probability $1/9$) then $\frac{6}{8}$ and then $\frac{1}{7}$

4 times: if 1st choice is 2 (Prob. $1/9$) then $\frac{4}{8}$ for 2nd choice and then $1/7$

once: if 1st choice is 3 (prob $1/9$) then $\frac{8}{8}$ for 2nd choice and then $1/7$

Total: $4 \times \frac{1}{9} \times \frac{6}{8} \times \frac{1}{7} + 4 \times \frac{1}{9} \times \frac{4}{8} \times \frac{1}{7} + 1 \times \frac{1}{9} \times \frac{8}{8} \times \frac{1}{7} = \frac{2}{21}$ $\boxed{B}$

The probability that a particular type of seedling produces red flowers is $\frac{1}{5}$. Eight of these seedlings are planted.

a) Write an expression for the probability that none of the eight seedlings produces red flowers.

b) Write an expression for the probability that at least one of the eight seedlings produces red flowers.

a) $\left(\frac{4}{8}\right)^5$

b) P(at least one of the seedlings produces red flowers) = 1 − P(none has red flowers)

$= 1 - \left(\frac{4}{5}\right)^8$