The **auxiliary angle** method of solving trigonometric equations involves changing an equation of the form $a\sin x \pm b\cos x = c$ into the form $r\sin(x \pm \alpha) = c$, which is then easier to solve. In this form, α is called the auxiliary angle.

This method can also be used to change $a\cos x \pm b\sin x = c$ into the form $r\cos(x \mp \alpha) = c$. In both cases, the constants a, b, r and α are positive real numbers.

For example, to express $a\sin x + b\cos x$ in the form $r\sin(x+\alpha)$:

$$\begin{aligned} \text{Let } a\sin x + b\cos x &= r\sin(x+\alpha) \\ &= r(\sin x\cos\alpha + \cos x\sin\alpha) \\ &= r\sin x\cos\alpha + r\cos x\sin\alpha \end{aligned}$$

This is an identity, so the coefficients of $\sin x$ and $\cos x$ on each side must be the same.

$$\begin{aligned} \text{i.e. } \quad a &= r\cos\alpha \\ b &= r\sin\alpha \\ \therefore \quad a^2 + b^2 &= r^2(\cos^2\alpha + \sin^2\alpha) \end{aligned}$$

Hence: $r^2 = a^2 + b^2$

$r = \sqrt{a^2+b^2}$ because r is a positive real number.

From the coefficients, there is also $\cos\alpha = \dfrac{a}{r}$ and $\sin\alpha = \dfrac{b}{r}$.

As a and b are positive constants, so $\cos\alpha$ and $\sin\alpha$ are also positive. This also means that α is in the first quadrant (i.e. it is an acute angle), such that $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{b}{a}$.

Hence, the auxiliary angle method gives: $a\sin x + b\cos x = r\sin(x+\alpha)$

which then allows you to obtain:

$$\begin{aligned} a\sin x - b\cos x &= r\sin(x-\alpha) \\ a\cos x + b\sin x &= r\cos(x-\alpha) \\ a\cos x - b\sin x &= r\cos(x+\alpha) \end{aligned}$$

In each case, $r = \sqrt{a^2+b^2}$ and α is an angle in the first quadrant such that $\tan\alpha = \dfrac{b}{a}$.

Example 1a: Express $\sqrt{3}\sin x - \cos x$ in the form $r\sin(x-\alpha)$

Solution

$$\begin{aligned} \sqrt{3}\sin x - \cos x &= r\sin(x-\alpha) \\ &= r(\sin x\cos\alpha - \cos x\sin\alpha) \\ &= r\sin x\cos\alpha - r\cos x\sin\alpha \end{aligned}$$

Equate coefficients of $\sin x$ and $\cos x$: $r\cos\alpha = \sqrt{3}$ [1]

$r\sin\alpha = 1$ [2]

$[1]^2 + [2]^2$: $r^2(\cos^2\alpha + \sin^2\alpha) = 4$

$r^2 = 4$

$r = 2$ (as $r > 0$)

Hence from [1] and [2]: $\cos\alpha = \dfrac{\sqrt{3}}{2}$ and $\sin\alpha = \dfrac{1}{2}$

As $\cos\alpha$ and $\sin\alpha$ are both positive, α is in the first quadrant, such that $\tan\alpha = \dfrac{1}{\sqrt{3}}$, i.e. $\alpha = \dfrac{\pi}{6}$

From the first equation: $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \dfrac{\pi}{6}\right)$

Example 1b: Express $3\cos x - 4\sin x$ in the form $r\cos(x+\alpha)$

Solution

$3\cos x - 4\sin x = r\cos(x+\alpha)$

$a = 3, b = 4$: $r = \sqrt{3^2+4^2} = 5$

$\tan\alpha = \frac{4}{3}$: $\alpha = 53°8'$

$\therefore\ 3\cos x - 4\sin x = 5\cos(x + 53°8')$

The two examples above illustrate two different auxiliary angle methods that may be used. You should practise both.

Important uses of the auxiliary angle method

- Writing $a\sin x + b\cos x$ in the form $r\sin(x+\alpha)$ tells you that the greatest and least values of the function are r and $-r$ respectively. This makes sketching functions like $y = a\sin x + b\cos x$ much easier.
- Writing $a\sin x + b\cos x$ in the form $r\sin(x+\alpha)$ allows you to solve equations of the type $a\sin x \pm b\cos x = c$.

Example 2

Sketch the graph of $y = \sqrt{3}\sin x - \cos x$, $0 \le x \le 2\pi$.

Solution

Example 1 **(a)** has already shown that $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$. Hence: $y = 2\sin\left(x - \frac{\pi}{6}\right)$

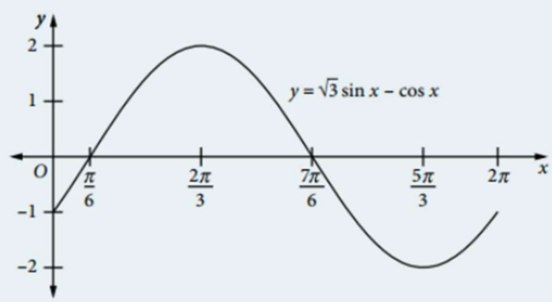
At the endpoints of the domain, $x = 0$ and $x = 2\pi$: $y = 2\sin\left(-\frac{\pi}{6}\right) = 2\sin\left(2\pi - \frac{\pi}{6}\right) = -1$

The greatest value of y is 2, where:

$$1 = \sin\left(x - \frac{\pi}{6}\right)$$
$$x - \frac{\pi}{6} = \frac{\pi}{2}$$
$$x = \frac{2\pi}{3}$$

The least value of y is -2, where:

$$-1 = \sin\left(x - \frac{\pi}{6}\right)$$
$$x - \frac{\pi}{6} = \frac{3\pi}{2}$$
$$x = \frac{5\pi}{3}$$



The graph crosses the x-axis where: $\sin\left(x - \frac{\pi}{6}\right) = 0$

$$x - \frac{\pi}{6} = 0, \pi$$
$$x = \frac{\pi}{6}, \frac{7\pi}{6}$$

Example 3

Solve the following equations.

(a) $\sqrt{3}\sin x - \cos x = 1,\ 0 \le x \le 2\pi$

(b) $8\cos x + 6\sin x = -3,\ 0° \le x \le 360°$

Solution

Method 1

(a) Example 1 **(a)** has already shown that $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$.

$$\therefore \quad 2\sin\left(x-\frac{\pi}{6}\right)=1$$

$$\sin\left(x-\frac{\pi}{6}\right)=\frac{1}{2}$$

$$x-\frac{\pi}{6}=\frac{\pi}{6},\ \frac{5\pi}{6}$$

$$x=\frac{\pi}{3},\ \pi$$

(b) Use $a\cos x + b\sin x = r\cos(x-\alpha)$.

$$8\cos x + 6\sin x = -3$$

$$a=8,\ b=6: \quad r=\sqrt{8^2+6^2}=10$$

$$\tan\alpha=\frac{6}{8}=0.75 \text{ so } \alpha=36°\,52'$$

$$\therefore\ 10\cos(x-36°\,52')=-3$$

$$\cos(x-36°\,52')=-0.3$$

$$x-36°\,52'=107°\,27',\ 252°\,33'$$

$$x=144°\,19',\ 289°\,25'$$

Method 2

You can express $\sin x$ and $\cos x$ in terms of $\tan\frac{x}{2}$ for all values of x, except $x=\pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ (because $\tan\frac{\pi}{2}$ is undefined for those values).

The t formulae (see Chapter 4) give $\sin x=\frac{2t}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2}$.

(a) $\sqrt{3}\sin x - \cos x = 1,\ 0 \le x \le 2\pi$: $\frac{2\sqrt{3}t}{1+t^2}-\frac{1-t^2}{1+t^2}=1$, where $t=\tan\frac{x}{2}$

$$2\sqrt{3}t-\left(1-t^2\right)=1+t^2$$

$$2\sqrt{3}t-1+t^2=1+t^2$$

$$\sqrt{3}t=1$$

$$t=\frac{1}{\sqrt{3}}$$

$$\therefore \quad \frac{x}{2}=\frac{\pi}{6} \text{ for } 0\le\frac{x}{2}\le\pi$$

$$x=\frac{\pi}{3} \text{ for } 0\le x\le 2\pi$$

Because $t=\tan\frac{x}{2}$ is undefined at $x=\pi$, you must now separately test whether $x=\pi$ is a solution.

$x=\pi$: LHS $=\sqrt{3}\sin\pi - \cos\pi = 0-(-1)=1=$ RHS

Hence $x=\pi$ is also a solution. The complete solution is $x=\frac{\pi}{3},\ \pi$.

(b) $8\cos x + 6\sin x = -3,\ 0° \le x \le 360°$: $\frac{8\left(1-t^2\right)}{1+t^2}+\frac{12t}{1+t^2}=-3$

$$8-8t^2+12t=-3-3t^2$$

$$5t^2-12t-11=0$$

$$t=\frac{12\pm\sqrt{144+220}}{10}$$

$$\tan\frac{x}{2}=3.108,\ -0.708 \quad \text{(to 3 d.p.)}$$

$$\frac{x}{2}=72°\,10',\ 144°\,42' \quad \text{for } 0°\le\frac{x}{2}\le 180°$$

$$x=144°\,20',\ 289°\,24' \quad \text{for } 0°\le x\le 360°$$

(Note the slight difference in the answers due to the rounding error when solving the quadratic equation.)

Because $t = \tan\frac{x}{2}$ is undefined at $x = \pi$, you must now separately test whether $x = 180°$ is a solution.

$x = 180°$: LHS $= 8\cos\pi + 6\sin\pi = -8 + 0 = -8 \neq$ RHS

Hence $x = 180°$ is not a solution of the equation.

Important note:

If you use the t formulae substitution to solve equations of the type $a\cos x + b\sin x = c$, you must also test to see whether $x = \pm n\pi$ is a solution of the equation. The use of the t formulae to solve a variety of equations will be covered later in this chapter.