HALF-ANGLE FORMULAE - THE t FORMULAE

1 If $t = \tan\frac{A}{2}$, then $\sin A + \cos A = \ldots$

$\boxed{A}$ $\frac{1+2t-t^2}{1+t^2}$ B $\frac{t^2-2t+1}{1+t^2}$ C $\frac{(1-t)^2}{1+t^2}$ D $\frac{(1+t)^2}{1+t^2}$

$$\sin A + \cos A = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \frac{2t - t^2 + 1}{1+t^2}$$

Response $\boxed{A}$

2 Simplify:

(a) $\frac{2\tan 9°}{1-\tan^2 9°}$ (b) $\frac{1-\tan^2 15°}{1+\tan^2 15°}$ (c) $\frac{1+\tan^2 22.5°}{2\tan 22.5°}$

a) $$\frac{2\tan 9}{1-\tan^2 9} = \tan(2\times 9) = \tan 18$$

b) $$\frac{1-\tan^2 15}{1+\tan^2 15} = \frac{1-t^2}{1+t^2} \quad \text{with } t = \tan 15$$

$\therefore$, as $\cos A = \frac{1-t^2}{1+t^2}$ with $t = \tan\frac{A}{2}$, then:

$$\frac{1-\tan^2 15}{1+\tan^2 15} = \cos 30 = \frac{\sqrt{3}}{2}$$

c) $$\frac{1+\tan^2 22.5}{2\tan 22.5} = \frac{1+t^2}{2t} \quad \text{with } t = \tan 22.5$$

But $\sin A = \frac{2t}{1+t^2}$ with $t = \tan\frac{A}{2}$, therefore:

$$\frac{1+\tan^2 22.5}{2\tan 22.5} = \frac{1}{\sin(2\times 22.5)} = \frac{1}{\sin 45} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

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3 If $t = \tan\frac{A}{2}$, express each of the following in terms of t:

(a) $\sin A - \cos A$ (b) $3\sin A + 4\cos A$ (c) $2\cos A - \sin A$ (d) $\cot A$

a) $\sin A - \cos A = \dfrac{2t}{1+t^2} - \dfrac{1-t^2}{1+t^2} = \dfrac{t^2+2t-1}{1+t^2}$

b) $3\sin A + 4\cos A = 3 \times \left(\dfrac{2t}{1+t^2}\right) + 4 \times \left(\dfrac{1-t^2}{1+t^2}\right)$

$\_\_\_\_\_\_ = \dfrac{6t + 4 - 4t^2}{1+t^2} = \dfrac{2(-2t^2+3t+2)}{1+t^2}$

c) $2\cos A - \sin A = 2 \times \left(\dfrac{1-t^2}{1+t^2}\right) - \dfrac{2t}{1+t^2}$

$\_\_\_\_\_\_ = \dfrac{-2t^2 - 2t + 2}{1+t^2} = \dfrac{-2(t^2+t-1)}{1+t^2}$

d) $\cot A = \dfrac{\cos A}{\sin A} = \dfrac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \dfrac{1-t^2}{2t}$

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3 If $t = \tan\frac{A}{2}$, express each of the following in terms of t:

(e) $\cot A - \tan A$ (f) $\dfrac{\cot A - \tan A}{\cot A + \tan A}$ (g) $1 - \frac{1}{2}\sin A \tan\frac{A}{2}$ (h) $1 + \tan A \tan\frac{A}{2}$

e) $$\cot A - \tan A = \frac{1}{\tan A} - \tan A = \frac{1}{\frac{2t}{1-t^2}} - \frac{2t}{1-t^2}$$

$$\text{———} = \frac{1-t^2}{2t} - \frac{2t}{1-t^2} = \frac{(1-t^2)^2 - (2t)^2}{2t(1-t^2)}$$

$$\text{———} = \frac{1-2t^2+t^4-4t^2}{2t(1-t^2)} = \frac{t^4-6t^2+1}{2t(1-t^2)}$$

f) As we have just done the numerator, we'll simplify the denominator first: $\cot A + \tan A = \dfrac{1}{\tan A} + \tan A = \dfrac{1}{\frac{2t}{1-t^2}} + \dfrac{2t}{1-t^2}$

$$\cot A + \tan A = \frac{1-t^2}{2t} + \frac{2t}{1-t^2} = \frac{(1-t^2)^2+(2t)^2}{2t(1-t^2)} = \frac{t^4+2t^2+1}{2t(1-t^2)}$$

$$\therefore \frac{\cot A - \tan A}{\cot A + \tan A} = \frac{\frac{t^4-6t^2+1}{2t(1-t^2)}}{\frac{t^4+2t^2+1}{2t(1-t^2)}} = \frac{t^4-6t^2+1}{t^4+2t^2+1}$$

g) $$1 - \frac{1}{2}\sin A \tan\frac{A}{2} = 1 - \frac{1}{2} \times \frac{2t}{1+t^2} \times t = 1 - \frac{t^2}{1+t^2}$$

$$1 - \frac{1}{2}\sin A \tan\frac{A}{2} = \frac{1+t^2}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1+t^2-t^2}{1+t^2} = \frac{1}{1+t^2}$$

h) $$1 + \tan A \tan\frac{A}{2} = 1 + \frac{2t}{1-t^2} \times t = 1 + \frac{2t^2}{1-t^2}$$

$$1 + \tan A \tan\frac{A}{2} = \frac{1-t^2}{1-t^2} + \frac{2t^2}{1-t^2} = \frac{1-t^2+2t^2}{1-t^2} = \frac{1+t^2}{1-t^2}$$

HALF-ANGLE FORMULAE - THE t FORMULAE

3 If $t = \tan\frac{A}{2}$, express each of the following in terms of t:

(i) $\dfrac{\tan A - \tan\frac{A}{2}}{\cot\frac{A}{2} + \tan A}$ (j) $\cot\frac{A}{2} - 2\cot A$ (k) $\dfrac{1+\sin A+\cos A}{1+\sin A-\cos A}$ (l) $\dfrac{\sin A+\sin\frac{A}{2}}{1+\cos A+\cos\frac{A}{2}}$

i) $$\frac{\tan A - \tan(A/2)}{\cot(A/2) + \tan A} = \frac{\frac{2t}{1-t^2} - t}{\frac{1}{t} + \frac{2t}{(1-t^2)}} = \frac{\frac{2t - t(1-t^2)}{1-t^2}}{\frac{1-t^2+2t^2}{t(1-t^2)}}$$

$$= \frac{\frac{t^3+t}{1-t^2}}{\frac{1+t^2}{t(1-t^2)}} = \frac{t(1+t^2)}{1-t^2} \times \frac{t(1-t^2)}{(1+t^2)} = t^2$$

j) $$\cot\left(\frac{A}{2}\right) - 2\cot A = \frac{1}{t} - 2\times\left(\frac{1-t^2}{2t}\right) = \frac{1}{t} - \left(\frac{1-t^2}{t}\right)$$

$$= \frac{1}{t}\left[1 - 1 + t^2\right] = \frac{t^2}{t} = t$$

k) $$\frac{1+\sin A+\cos A}{1+\sin A-\cos A} = \frac{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} = \frac{1+t^2+2t+1-t^2}{1+t^2+2t-1+t^2}$$

$$= \frac{2t+2}{2t^2+2t} = \frac{t+1}{t(t+1)} = \frac{1}{t}$$

l) $$\frac{\sin A + \sin(A/2)}{1+\cos A+\cos(A/2)} = \frac{2\sin(A/2)\cos(A/2) + \sin(A/2)}{2\cos^2(A/2) + \cos(A/2)}$$

$$= \frac{\sin(A/2)\left[2\cos(A/2)+1\right]}{\cos(A/2)\left[2\cos(A/2)+1\right]}$$

$$= \tan(A/2) = t$$

6 If $t = \tan\frac{A}{2}$, solve for t the equation $12\tan A = 5$, $180° < A < 270°$.

$$12\tan A = 5 \iff 12 \times \frac{2t}{1-t^2} = 5$$

$$\iff 24t = 5 - 5t^2$$

$$\iff 5t^2 + 24t - 5 = 0$$

$$\Delta = 24^2 - 4 \times (-5) \times 5 = 676 = 26^2$$

So 2 solutions $t = \frac{-24+26}{10} = \frac{1}{5}$ or $t = \frac{-24-26}{10} = -5$

So $t = \frac{1}{5}$ or $t = -5$

But $180° < A < 270°$ so $90 < \frac{A}{2} < 135$

so $\left(\frac{A}{2}\right)$ is in the II quadrant, i.e. $\sin(A/2) > 0$ and $\cos(A/2) < 0$

so $\tan(A/2)$ must be negative.

So $t = \frac{1}{5}$ is impossible

$t = -5$ is the only possible solution.

8 If $\sec\theta - \tan\theta = x$, prove that $x = \dfrac{1-t}{1+t}$.

$$x = \sec\theta - \tan\theta = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \frac{1-\sin\theta}{\cos\theta}$$

$$x = \frac{1 - \frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}} = \frac{\frac{1+t^2-2t}{1+t^2}}{\frac{1-t^2}{1+t^2}} = \frac{1+t^2-2t}{1-t^2}$$

$$x = \frac{(1-t)^2}{(1-t)(1+t)} = \frac{1-t}{1+t}$$