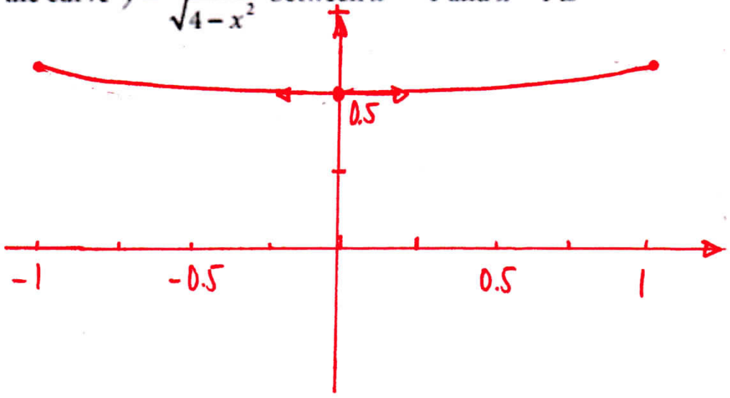
5 Find the volume of the solid generated when the part of the curve $y = \frac{1}{\sqrt{4-x^2}}$ between $x = -1$ and $x = 1$ is rotated about the x-axis.



This volume is $\pi \int_{-1}^{1} \left(\frac{1}{\sqrt{4-x^2}}\right)^2 dx$

$$V = \pi \int_{-1}^{1} \frac{1}{4-x^2}\, dx$$

$$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)} = \frac{a}{2-x} + \frac{b}{2+x} = \frac{x(a-b)+(2a+2b)}{4-x^2}$$

So $\begin{cases} a = b \\ 2a+2b = 1 \end{cases} \Leftrightarrow \begin{cases} a = b \\ 2a+2a = 1 \end{cases} \Leftrightarrow \begin{cases} a = 1/4 \\ b = 1/4 \end{cases}$

$$V = \frac{\pi}{4} \int_{-1}^{1} \left[\frac{1}{2-x} + \frac{1}{2+x}\right] dx$$

$$V = \frac{\pi}{4} \Big[-\ln|2-x| + \ln|2+x| \Big]_{-1}^{1}$$

$$V = \frac{\pi}{4} \left[\ln\left|\frac{2+x}{2-x}\right| \right]_{-1}^{1}$$

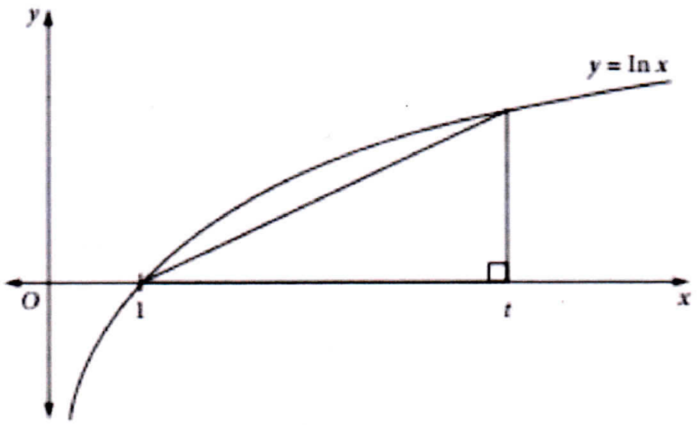
$$V = \frac{\pi}{4} \left[\ln\left(\frac{3}{1}\right) - \ln\left(\frac{1}{3}\right) \right] = \frac{\pi}{4} \left[\ln 3 + \ln 3 \right]$$

So $V = \dfrac{\pi \ln 3}{2}$ units^2.

7 The diagram shows the graph of $y = \ln x$.

(a) Find $\int_1^t \ln x \, dx$.

(b) By comparing relevant areas in the diagram, or otherwise, show that $\ln t > 2\left(\frac{t-1}{t+1}\right)$, for $t > 1$.



a) $\frac{d}{dx}(x \ln x) = \ln x + 1$

So $\ln x = \frac{d}{dx}(x \ln x) - 1$

So $\int_1^t \ln x \, dx = [x \ln x]_1^t - \int_1^t dx$

$\int_1^t \ln x \, dx = t \ln t - (t-1) = t \ln t - t + 1$

b) The curve $f(x) = \ln x$ is above the line shown on the diagram, which equation is $y - 0 = \frac{\ln t}{t-1} \times (x-1)$

or $y = \left(\frac{\ln t}{t-1}\right) x - \left(\frac{\ln t}{t-1}\right)$.

$\therefore \int_1^t \ln x \, dx > \frac{1}{2}(t-1) \times \ln t$ (area of triangle).

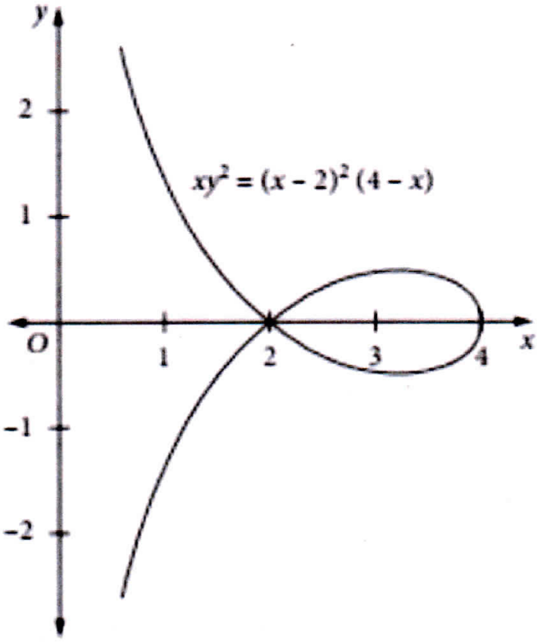
$\therefore t \ln t - t + 1 > \frac{1}{2}(t-1) \ln t$

$\Leftrightarrow \ln t \left[t - \frac{1}{2}t + \frac{1}{2}\right] > -1 + t$

$\Leftrightarrow \ln t \times \left[\frac{t+1}{2}\right] > t - 1$

$\Leftrightarrow \ln t > \frac{2(t-1)}{t+1}$

10 The diagram shows the graph of $xy^2 = (x-2)^2(4-x)$. Find the volume of the solid formed by rotating the loop in the graph of $xy^2 = (x-2)^2(4-x)$ about the x-axis.



The curve cuts the x-axis at $x = 2$ and $x = 4$

$\therefore$ this volume is

$$\pi \int_2^4 (f(x))^2 \, dx$$

$$xy^2 = (x-2)^2(4-x)$$

$$\Leftrightarrow y^2 = \frac{(x-2)^2(4-x)}{x}$$

$$\text{So } V = \pi \int_2^4 \frac{(x-2)^2(4-x)}{x} \, dx$$

$$(x-2)^2(4-x) = (x^2-4x+4)(4-x) = -x^3 + x^2(4+4) + x(-16-4) + 16$$

$$= -x^3 + 8x^2 - 20x + 16 \quad , \text{ So}$$

$$V = \pi \int_2^4 \left[-x^2 + 8x - 20 + \frac{16}{x} \right] dx$$

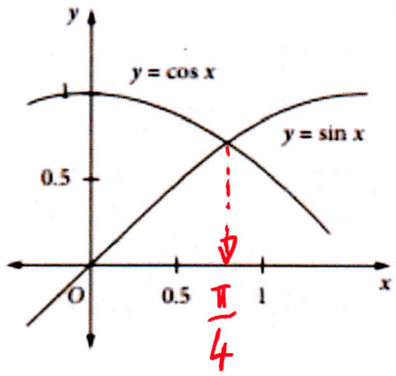
$$V = \pi \left[-\frac{x^3}{3} + 4x^2 - 20x + 16 \ln x \right]_2^4$$

$$V = \pi \left[\left(-\frac{4^3}{3} + 4\times16 - 20\times4 + 16\ln 4 \right) - \left(-\frac{2^3}{3} + 4\times4 - 20\times2 + 16\ln 2 \right) \right]$$

$$V = \pi \left[\left(-\frac{112}{3} + 32\ln 2 \right) - \left(-\frac{80}{3} + 16\ln 2 \right) \right]$$

$$V = \pi \left[-\frac{32}{3} + 16\ln 2 \right] \simeq 1.33 \text{ units}^3$$

11 (a) The region bounded by the curves $y = \sin x$, $y = \cos x$ and the y-axis is rotated about the x-axis. Calculate the volume of the solid of revolution formed.



$$V = \pi \int_0^{\pi/4} \cos^2 x \, dx - \pi \int_0^{\pi/4} \sin^2 x \, dx$$

$$V = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) \, dx$$

$$V = \pi \int_0^{\pi/4} \cos 2x \, dx$$

$$V = \pi \left[\frac{\sin 2x}{2} \right]_0^{\pi/4}$$

$$V = \pi \frac{\sin(\pi/2)}{2} = \frac{\pi}{2} \text{ units}^2$$

12 When the circle $x^2+y^2=a^2$ is rotated about the line $x=b$ $(b>a)$ to generate a torus, the volume of the solid of revolution formed is given by $V=4\pi\int_{-a}^{a}(b-x)\sqrt{a^2-x^2}\,dx$. Calculate this volume.

$$V=4\pi\int_{-a}^{a}(b-x)\sqrt{a^2-x^2}\,dx$$

let $x=a\sin\theta$ so $\frac{dx}{d\theta}=a\cos\theta$ $\quad dx=a\cos\theta\,d\theta$

when $x=-a$ $\theta=-\pi/2$; when $x=a$ $\theta=\pi/2$

$$\text{So } V=4\pi\int_{-\pi/2}^{\pi/2}(b-a\sin\theta)\sqrt{a^2-a^2\sin^2\theta}\times a\cos\theta\,d\theta$$

$$V=4\pi\int_{-\pi/2}^{\pi/2}(b-a\sin\theta)\times a\times\cos\theta\times a\cos\theta\,d\theta$$

$$V=4\pi a^2\int_{-\pi/2}^{\pi/2}\left[b\cos^2\theta-a\sin\theta\cos^2\theta\right]d\theta$$

if we consider $f(\theta)=\sin\theta\cos^2\theta$

$f(-\theta)=\sin(-\theta)\cos^2(-\theta)=-\sin\theta\cos^2\theta=-f(\theta)$ so f is an odd function

$\therefore\ \forall a\in\mathbb{R}\quad\int_{-a}^{a}f(\theta)\,d\theta=0$, so the 2nd part is zero.

$$V=4\pi a^2b\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$V=4\pi a^2b\int_{-\pi/2}^{\pi/2}\left[\frac{\cos2\theta+1}{2}\right]d\theta=\frac{4\pi a^2b}{2}\left[\frac{\sin2\theta}{2}+\theta\right]_{-\pi/2}^{\pi/2}$$

$$V=2\pi a^2b\left[\left(0+\frac{\pi}{2}\right)-\left(0+\left(\frac{-\pi}{2}\right)\right)\right]=2\pi a^2b\times\pi$$

So $V=2\pi^2a^2b$ units^3

14 Find the particular solution of $\frac{dy}{dx} = x^2 \sin x + 2x\cos x - 2\sin x$, given that $y = 6$ when $x = 0$.

$y = \int (x^2 \sin x + 2x\cos x - 2\sin x)\,dx$

$y = \int x^2 \sin x\,dx + 2\int x\cos x\,dx - 2\int \sin x\,dx$

$y = +2\cos x + 2\int x\cos x\,dx + \int x^2 \sin x\,dx$

integration by parts: $u(x) = x$ $\quad v(x) = \sin x$

$u'(x) = 1 \quad v'(x) = \cos x$

$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

For the last one, we also integrate by parts.

$u(x) = x^2 \quad v(x) = -\cos x$

$u'(x) = 2x \quad v'(x) = \sin x$

So $\int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$

$= -x^2\cos x + 2[x\sin x + \cos x] + C$

So $y = 2\cos x + 2[x\sin x + \cos x] + [-x^2\cos x + 2x\sin x + 2\cos x] + C$

$y = -x^2\cos x + 4x\sin x + 6\cos x + C$

When $x = 0$ $\quad y(0) = 0 + 0 + 6 + C = 6$

so $C = 0$ for that particular solution, which is

$y = -x^2\cos x + 4x\sin x + 6\cos x.$

15 Find the particular solution of $\frac{dy}{dx} = e^{-x} \sin x$, given that $y = -\frac{1}{2}$ when $x = 0$.

$y = \int e^{-x} \sin x \, dx$ we integrate by parts. LIATE

$u(x) = \sin x$ $\quad v(x) = -e^{-x}$

$u'(x) = \cos x$ $\quad v'(x) = e^{-x}$

$\int e^{-x} \sin x \, dx = -e^{-x} \sin x + \int e^{-x} \cos x \, dx$ another integration by parts.

$u(x) = \cos x$ $\quad v(x) = -e^{-x}$

$u'(x) = -\sin x$ $\quad v'(x) = e^{-x}$

So $\int e^{-x} \sin x \, dx = -e^{-x} \sin x + \left[-e^{-x} \cos x - \int e^{-x} \sin x \, dx \right]$

So $\int e^{-x} \sin x \, dx = -e^{-x} (\sin x + \cos x) - \int e^{-x} \sin x \, dx$

$\therefore 2 \int e^{-x} \sin x \, dx = -e^{-x} (\sin x + \cos x) + C$

$\therefore \int e^{-x} \sin x \, dx = -\frac{e^{-x}}{2} (\sin x + \cos x) + C$

$\therefore y = -\frac{e^{-x}}{2} (\cos x + \sin x) + C$

$y(0) = -\frac{e^{-0}}{2} (\cos 0 + \sin 0) + C = -\frac{1}{2} \times 1 + C$

So $C - \frac{1}{2} = -\frac{1}{2}$ so $C = 0$

$y(x) = -\frac{e^{-x}}{2} (\sin x + \cos x)$

USES OF INTEGRATION

19 **(a)** The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is rotated about the x-axis. Show that the volume of the solid generated is $\frac{1}{3}4\pi ab^2$ cubic units.

(b) When the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is rotated about the line $x=a$, the volume of the solid of revolution is given by $V=2\pi\int_{-a}^{a}2y(a-x)dx$. Calculate the volume of this solid of revolution.

a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \iff \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \iff y^2=b^2-\frac{b^2x^2}{a^2}$

So $V=\pi\int_{-a}^{a}y^2\,dx=\pi\int_{-a}^{a}\left(b^2-\frac{b^2x^2}{a^2}\right)dx$

$V=2\pi\int_0^a\left(b^2-\frac{b^2x^2}{a^2}\right)dx=2\pi b^2\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$

$V=2\pi b^2\left[x-\frac{x^3}{3a^2}\right]_0^a$

$V=2\pi b^2\left[a-\frac{a^3}{3a^2}\right]=2\pi b^2\left(a-\frac{a}{3}\right)=\frac{4\pi}{3}ab^2$

b) $V=2\pi\int_{-a}^{a}2y(a-x)\,dx$ $\qquad y=\sqrt{b^2-\frac{b^2x^2}{a^2}}=b\sqrt{1-\frac{x^2}{a^2}}$

$V=4\pi b\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}\,(a-x)\,dx$

let $x=a\sin\theta$ $\qquad \frac{dx}{d\theta}=a\cos\theta$ $\qquad dx=a\cos\theta\,d\theta$

when $x=a$ $\quad \theta=\pi/2$ $\qquad$ when $x=-a$ $\quad \theta=-\pi/2$

$V=4\pi ba^2\int_{-\pi/2}^{\pi/2}\cos\theta\,(1-\sin\theta)\cos\theta\,d\theta$

$V=4\pi a^2b\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta-\underbrace{\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta\,d\theta}_{=0 \text{ as odd function}}\right]$

$V=4\pi a^2b\left[\int_{-\pi/2}^{\pi/2}\frac{\cos2\theta+1}{2}\,d\theta\right]=\frac{4\pi a^2b}{2}\left[\frac{\sin2\theta}{2}+\theta\right]_{-\pi/2}^{\pi/2}$

$V=2\pi a^2b\left[\left(0+\frac{\pi}{2}\right)-\left(0-\frac{\pi}{2}\right)\right]=2\pi^2a^2b$ units^3

22 **(a)** Let $I_n = \int_1^2 (\log_e x)^n \, dx$ where n is a positive integer. Show that $I_n = 2(\log_e 2)^n - nI_{n-1}$.

(b) Let $f(x) = (\log_e x)^n$ where integer $n \geq 2$. By considering the first and second derivatives, show that $f(x)$ is increasing and concave up over the domain $1 < x \leq 2$. Sketch the curve $y = (\log_e x)^n$ (where $n \geq 2$) over the domain $1 \leq x \leq 2$.

(c) By considering the area of a triangle that acts as an upper bound, show that: $nI_{n-1} > \frac{3}{2}(\log_e 2)^n$.

(d) Hence show that $\frac{2}{3} < \log_e 2 < 2$.

a) $I_n = \int_1^2 1 \times (\ln x)^n \, dx$ we integrate by parts.

$u(x) = (\ln x)^n$ $\qquad$ $v(x) = x$

$u'(x) = \dfrac{n \times (\ln x)^{n-1}}{x}$ $\qquad$ $v'(x) = 1$

$\therefore \int (\ln x)^n dx = x \times (\ln x)^n - \int \dfrac{n (\ln x)^{n-1}}{x} \times x \, dx$

$\therefore \int (\ln x)^n dx = x \times (\ln x)^n - n \int (\ln x)^{n-1} dx$

$\therefore \int_1^2 (\ln x)^n dx = \left[x \times (\ln x)^n \right]_1^2 - n \int_1^2 (\ln x)^{n-1} dx$

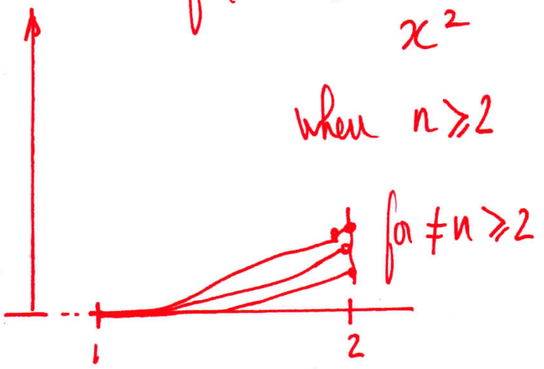
$I_n = 2(\ln 2)^n - n I_{n-1}$

b) $f(x) = (\ln x)^n$ $\quad$ then $f'(x) = \dfrac{n (\ln x)^{n-1}}{x}$ $\quad$ $f'(x) > 0$ when $1 < x \leq 2$ so f is increasing.

$f''(x) = \dfrac{n \times (n-1)(\ln x)^{n-2} x - n (\ln x)^{n-1}}{x^2}$

$f''(x) = \dfrac{n (\ln x)^{n-2}}{x^2} \left[(n-1)x - \ln x \right]$

when $n \geq 2$ $\quad$ $(n-1)x > \ln x$ $\quad$ so $f''(x) > 0$ so f concave up



c) $f(x) = (\ln x)^n$ For $x = 2$ $f(2) = (\ln 2)^n$

as $\ln 2 < 1$ ($\simeq 0.69$) the maximum value would be for $n = 2$

so is $0.48... = (\ln 2)^2$

$$I_n < \frac{1}{2} \times (2-1) \times (\ln 2)^n$$

So $I_n < \frac{(\ln 2)^n}{2}$ But $I_n = 2(\ln 2)^n - n I_{n-1}$

$$\therefore \quad 2(\ln 2)^n - n I_{n-1} < \frac{(\ln 2)^n}{2}$$

$$\therefore \quad n I_{n-1} > 2(\ln 2)^n - \frac{(\ln 2)^n}{2}$$

$$n I_{n-1} > (\ln 2)^n \left[2 - \frac{1}{2}\right]$$

$$\therefore \quad n I_{n-1} > \frac{3}{2}(\ln 2)^n \qquad \text{Equation } ①$$

d) For $n = 2$ $2 I_{2-1} > \frac{3}{2}(\ln 2)^2$

or $4 I_1 > 3(\ln 2)^2$

$$I_1 = \int_1^2 \ln x \, dx = \left[x \ln x - x\right]_1^2 = (2\ln 2 - 2) - (-1) = 2\ln 2 - 1$$

So $8\ln 2 - 4 > 3(\ln 2)^2$ or $3(\ln 2)^2 - 8\ln 2 + 4 < 0$

Change of variable $x = \ln 2$ so $3x^2 - 8x + 4 < 0$

$\Delta = 8^2 - 4 \times 4 \times 3 = 16 = 4^2$ so 2 roots

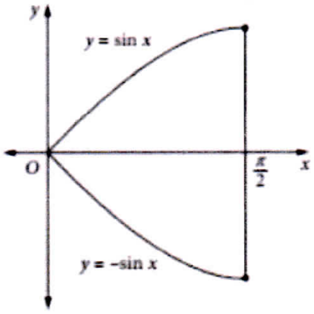
$x_1 = \frac{+8-4}{6} = \frac{4}{6} = \frac{2}{3}$ and $x_2 = \frac{8+4}{6} = 2.$

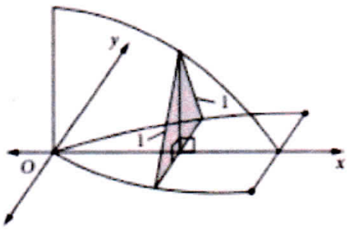
$f(x) = 3x^2 - 8x + 4$ is a parabola concave up.

For the value to be negative, we must have $\frac{2}{3} < x < 2$

i.e $\frac{2}{3} < \ln 2 < 2$

26 The base of a solid is formed by the area bounded by $y = \sin x$ and $y = -\sin x$ for $0 < x < \frac{\pi}{2}$ and the line $x = \frac{\pi}{2}$. Vertical cross-sections of the solid taken parallel to the y-axis are isosceles triangles with equal sides of length 1 unit, as shown. The volume of this solid is given by $V = \int_0^{\frac{\pi}{2}} y\sqrt{1-y^2}\,dx$ where $y = \sin x$. Calculate the exact volume of this solid.





$$V = \int_0^{\pi/2} y\sqrt{1-y^2}\,dx \qquad \text{where } y = \sin x.$$

$$V = \int_0^{\pi/2} \sin x \sqrt{1-\sin^2 x}\,dx = \int_0^{\pi/2} \sin x \cos x \, dx.$$

$$V = \int_0^{\pi/2} \frac{\sin 2x}{2}\,dx$$

$$V = \frac{1}{2}\int_0^{\pi/2} \sin 2x\,dx$$

$$V = \frac{1}{2}\left[\frac{-\cos 2x}{2}\right]_0^{\pi/2}$$

$$V = \frac{1}{4}\left[\cos 2x\right]_{\pi/2}^{0}$$

$$V = \frac{1}{4}\left[1 - \cos \pi\right]$$

$$V = \frac{1}{4}\left[1 - (-1)\right]$$

$$V = \frac{2}{4} = \frac{1}{2}$$

That was easy for the last one