PROJECTION OF VECTORS

1 For each of the following pairs of vectors, find the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ (b) $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ (c) $\underset{\sim}{a} = 8\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 8\underset{\sim}{j}$

a) $$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(4\vec{i} - \vec{j})\cdot(3\vec{i} + 4\vec{j})}{\sqrt{3^2 + 4^2}}$$

$$\text{---} = \frac{12 - 4}{\sqrt{9 + 16}} = \frac{8}{\sqrt{25}} = \frac{8}{5}$$

b) $$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(4\vec{i} + 3\vec{j})\cdot(3\vec{i} + 2\vec{j})}{\sqrt{3^2 + 2^2}}$$

$$\text{---} = \frac{4\times 3 + 3\times 2}{\sqrt{9 + 4}} = \frac{12 + 6}{\sqrt{13}} = \frac{18}{\sqrt{13}} = \frac{18\sqrt{13}}{13}$$

c) $$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(8\vec{i} + 3\vec{j})\cdot(-3\vec{i} + 8\vec{j})}{\sqrt{(-3)^2 + 8^2}}$$

$$\text{---} = \frac{-24 + 24}{\sqrt{---}} = 0$$

PROJECTION OF VECTORS

2 For each of the following pairs of vectors, find the vector projections of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ (b) $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ (c) $\underset{\sim}{a} = 8\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$

The vector projection of $\vec{a}$ onto $\vec{b}$ is the vector $\vec{c}$ such that

$\vec{c} = |\vec{c}| \dfrac{\vec{b}}{|\vec{b}|}$ But $|\vec{c}| = |\vec{a}| \cos\theta = |\vec{a}| \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$\text{So } \vec{c} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} \times \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{(|\vec{b}|)^2} \times \vec{b}$$

$$a)\ \vec{c} = \frac{12+6}{(3^2+2^2)} \times (3\vec{i} + 2\vec{j}) = \frac{18}{13}(3\vec{i} + 2\vec{j})$$

$$b)\ \vec{c} = \frac{(4\vec{i} - \vec{j})\cdot(3\vec{i} + 4\vec{j})}{(3^2+4^2)} \times [3\vec{i} + 4\vec{j}]$$

$$\vec{c} = \frac{12-4}{25} \times [3\vec{i} + 4\vec{j}] = \frac{8}{25}(3\vec{i} + 4\vec{j})$$

$$c)\ \vec{c} = \frac{[(8\vec{i} + 4\vec{j})\cdot(-3\vec{i} + 6\vec{j})]}{(-3)^2 + 6^2} \times [-3\vec{i} + 6\vec{j}]$$

$$\vec{c} = \frac{(-24+24)}{\text{—}} \times \text{—} = \vec{0}$$

3 For each of the following pairs of vectors, find the vector projections of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ **(c)** $\underset{\sim}{a} = 8\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$

a) The projection of $\vec{a}$ onto $\vec{b}$ is $\vec{c}$ such that $\vec{c} = |\vec{c}|\dfrac{\vec{b}}{|\vec{b}|}$

But $|\vec{c}| = |\vec{a}|\cos\theta = |\vec{a}|\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{12+6}{\sqrt{3^2+2^2}} = \dfrac{18}{\sqrt{13}}$

So $\vec{c} = \dfrac{18}{\sqrt{13}} \times \dfrac{(3\vec{i}+2\vec{j})}{\sqrt{13}} = \dfrac{18}{13}(3\vec{i}+2\vec{j}) = \dfrac{54}{13}\vec{i} + \dfrac{36}{13}\vec{j}$

Then $\vec{a} = \vec{c} + \vec{c_\perp}$ so $\vec{c_\perp} = \vec{a} - \vec{c} = \left(4 - \dfrac{54}{13}\right)\vec{i} + \left(3 - \dfrac{36}{13}\right)\vec{j}$

$\vec{c_\perp} = -\dfrac{2}{13}\vec{i} + \dfrac{3}{13}\vec{j}$

b) using finding from a) $\vec{c} = \dfrac{(\vec{a}\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}$

$\vec{c} = \dfrac{(12-4)(3\vec{i}+4\vec{j})}{(9+16)} = \dfrac{24\vec{i}+32\vec{j}}{25} = \dfrac{24}{25}\vec{i} + \dfrac{32}{25}\vec{j}$

Then $\vec{a} = \vec{c} + \vec{c_\perp}$ so $\vec{c_\perp} = \vec{a} - \vec{c} = \left(4 - \dfrac{24}{25}\right)\vec{i} + \left(-1 - \dfrac{32}{25}\right)\vec{j}$

$\vec{c_\perp} = \dfrac{76}{25}\vec{i} - \dfrac{57}{25}\vec{j}$

c) Again $\vec{c} = \dfrac{(\vec{a}\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}$

$\vec{c} = \dfrac{(-24+24)\,\vec{b}}{|\vec{b}|^2} = \vec{0}$

Then $\vec{a} = \vec{c} + \vec{c_\perp} = \vec{0} + \vec{c_\perp}$ so $\vec{c_\perp} = \vec{a} = 8\vec{i} + 4\vec{j}$

4 For the following vectors, find the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$

a) $\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|} = \dfrac{12+6}{\sqrt{4^2+3^2}} = \dfrac{18}{5}$

b) $\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|} = \dfrac{12-4}{\sqrt{4^2+1^2}} = \dfrac{8}{\sqrt{17}} = \dfrac{8\sqrt{17}}{17}$

6 For $\underset{\sim}{a} = -2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 2\underset{\sim}{j}$, the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is:

A $\frac{\sqrt{2}}{2}$ $\boxed{\textbf{B}}$ $\frac{-\sqrt{2}}{2}$ **C** $\frac{-2\sqrt{13}}{13}$ **D** $\frac{2\sqrt{13}}{13}$

$\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{4-6}{\sqrt{2^2+2^2}} = \dfrac{-2}{\sqrt{8}} = \dfrac{-2}{2\sqrt{2}} = -\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

7 The vector projection of $\overset{\vec{a}}{3\underset{\sim}{i} + 2\underset{\sim}{j}}$ onto $\overset{\vec{b}}{-\underset{\sim}{i} + 2\underset{\sim}{j}}$ is $\overset{\vec{c}}{\frac{1}{5}\left(-\underset{\sim}{i} + 2\underset{\sim}{j}\right)}$. What is the vector projection of $3\underset{\sim}{i} + 2\underset{\sim}{j}$ perpendicular to $-\underset{\sim}{i} + 2\underset{\sim}{j}$?

A $\frac{8}{5}\left(3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$ **B** $\frac{8}{5}\left(-\underset{\sim}{i} + 2\underset{\sim}{j}\right)$ $\boxed{\textbf{C}}$ $\frac{8}{5}\left(2\underset{\sim}{i} + \underset{\sim}{j}\right)$ **D** $2\underset{\sim}{i} + \underset{\sim}{j}$

$\vec{a} = \vec{c} + \vec{c}_\perp$ so $\vec{c}_\perp = \vec{a} - \vec{c}$

$\vec{c}_\perp = 3\vec{i} + 2\vec{j} - \left(\frac{1}{5}(-\vec{i}+2\vec{j})\right)$

$\vec{c}_\perp = \dfrac{16}{5}\vec{i} + \dfrac{8}{5}\vec{j} = \dfrac{8}{5}\left(2\vec{i} + \vec{j}\right)$

PROJECTION OF VECTORS

9 Consider two vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$.

(a) Find the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(b) Find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(c) Find the vector projection of $\underset{\sim}{a}$ perpendicular to the direction of $\underset{\sim}{b}$.

(d) Hence, express the vector $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ in terms of projections onto and perpendicular to $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$.

a) The scalar projection of $\vec{a}$ onto $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$, so

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{6+8}{\sqrt{2^2+2^2}} = \frac{14}{\sqrt{8}} = \frac{14}{2\sqrt{2}} = \frac{7}{\sqrt{2}} = \frac{7\sqrt{2}}{2}$$

b) The vector projection of $\vec{a}$ onto $\vec{b}$ is $\vec{c}$ such that.

$$\vec{c} = |\vec{c}|\,\frac{\vec{b}}{|\vec{b}|}$$

But $|\vec{c}| = |\vec{a}|\cos\theta = |\vec{a}|\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

so $$\vec{c} = \frac{\vec{a}\cdot\vec{b}}{(|\vec{b}|)^2} \times \vec{b} = \frac{(6+8)}{(2^2+2^2)} \times (2\vec{i} - 2\vec{j}) = \frac{14}{8}(2\vec{i} - 2\vec{j}) = \frac{7}{2}\vec{i} - \frac{7}{2}\vec{j}$$

c) $\vec{a} = \vec{c} + \vec{c_\perp}$ so $\vec{c_\perp} = \vec{a} - \vec{c}$

$$\vec{c_\perp} = \left(3\vec{i} - 4\vec{j}\right) - \left(\frac{7}{2}\vec{i} - \frac{7}{2}\vec{j}\right) = -\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j}$$

d) So $$\vec{a} = \left[\frac{7}{2}\vec{i} - \frac{7}{2}\vec{j}\right] + \left[-\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j}\right]$$

PROJECTION OF VECTORS

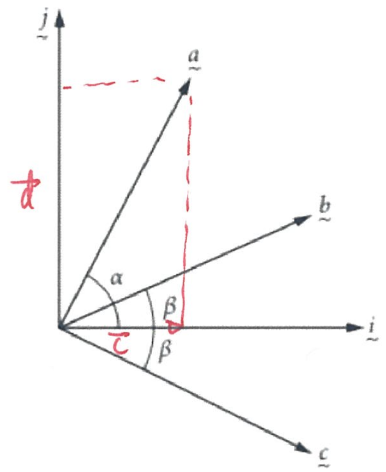
11 $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are unit vectors in the Cartesian plane.

(a) Show that $\underset{\sim}{a} = \cos\alpha\,\underset{\sim}{i} + \sin\alpha\,\underset{\sim}{j}$.

(b) Derive similar expressions for $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

(c) Find $\underset{\sim}{a} \bullet \underset{\sim}{b}$ and $\underset{\sim}{a} \bullet \underset{\sim}{c}$.

(d) Hence deduce the compound angle formulas $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.



a) $\vec{c} = |\vec{a}|\cos\alpha \times \vec{i} = \cos\alpha\,\vec{i}$

$\vec{d} = |\vec{a}|\sin\alpha \times \vec{j} = \sin\alpha\,\vec{j}$

$\vec{a} = \vec{c} + \vec{d} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$

b) likewise $\vec{b} = \cos\beta\,\vec{i} + \sin\beta\,\vec{j}$

$\vec{c} = \cos(-\beta)\,\vec{i} + \sin(-\beta)\,\vec{j} = \cos\beta\,\vec{i} - \sin\beta\,\vec{j}$

c) $\vec{a}.\vec{b} = [\cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}].[\cos\beta\,\vec{i} + \sin\beta\,\vec{j}]$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\vec{a}.\vec{c} = [\cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}].[\cos\beta\,\vec{i} - \sin\beta\,\vec{j}]$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

d) But $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos(\alpha - \beta) = \cos(\alpha - \beta)$

so $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

And $\vec{a}.\vec{c} = |\vec{a}||\vec{c}|\cos(\alpha + \beta) = \cos(\alpha + \beta)$

So $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$