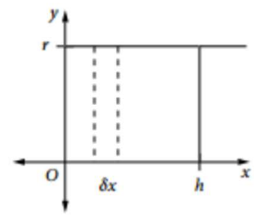
VOLUMES OF SOLIDS OF REVOLUTION

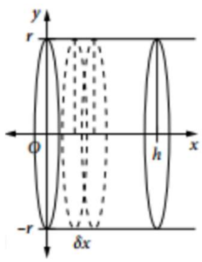


You have seen that the area of a region bounded by a line $y = r$, the x-axis and the ordinates $x = 0$ and $x = h$ can be found by adding up the areas of all the rectangles of width δx and height r between $x = 0$ and $x = h$, as δx becomes vanishingly small: $A = \lim_{\delta x \to 0} \sum_{0}^{h} f(x)\delta x$.

This area is given by the definite integral $A = \int_0^h r\,dx$, which is $A = \int_0^h r\,dx = [rx]_0^h = rh$.

You should recognise this as the area of a rectangle of sides r and h.

Consider what happens when the area bounded by $y = r$, the x-axis and the ordinates $x = 0$ and $x = h$ is rotated about the x-axis to form a solid of revolution, as shown in the diagram below to the left. The solid of revolution formed is a cylinder of radius r and height h.

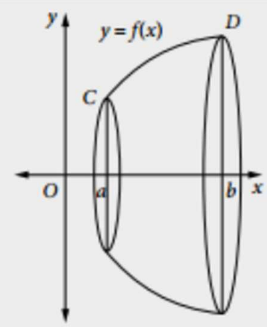


The rectangles of side r and width δx have become circular disks of radius r and thickness δx. The volume of this disk is given by $\Delta V = \pi(f(x))^2\,\delta x$. Adding all the disks as δx gets smaller gives $V = \lim_{\delta x \to 0} \sum_{0}^{h} \pi(f(x))^2\,\delta x$, which is given by the definite integral $V = \pi\int_0^h r^2\,dx$.

Thus the volume is $V = \pi\int_0^h r^2\,dx = \pi\left[r^2x\right]_0^h = \pi r^2 h$, which you should recognise as the volume of a cylinder of radius r and height h.

When the arc CD of the curve $y = f(x)$ on the interval $a \le x \le b$ is rotated about the x-axis, the volume of the solid of revolution formed is given by:

$$V = \pi\int_a^b (f(x))^2\,dx \qquad \textbf{or} \qquad V = \pi\int_a^b y^2\,dx$$

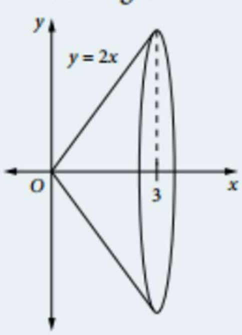


Example 1

Calculate the volume of the solid formed when the portion of the line $y = 2x$ between $x = 0$ and $x = 3$ is rotated about the x-axis. What is the name of the kind of solid formed?

Solution

Draw a diagram:



$$\begin{aligned}\text{Volume} &= \pi\int_a^b y^2\,dx \\ &= \pi\int_0^3 (2x)^2\,dx \\ &= 4\pi\int_0^3 x^2\,dx \\ &= 4\pi\left[\frac{x^3}{3}\right]_0^3 \\ &= 4\pi(9-0) \\ &= 36\pi \text{ units}^3\end{aligned}$$

The solid is a right circular cone of base radius 6 and height 3.

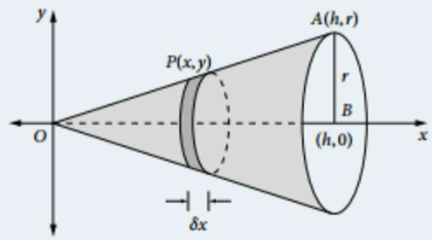
Example 2

Find the volume of a right circular cone of height h and base radius r.

Solution

The cone can be considered as a solid of revolution generated by rotating the right-angled triangle OAB about the x-axis.

The equation of OA is $y = \frac{rx}{h}$.



$$V = \pi \int_a^b y^2\,dx$$

$$V = \pi \int_0^h \frac{r^2x^2}{h^2}\,dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi r^2}{h^2} \times \frac{h^3}{3}$$

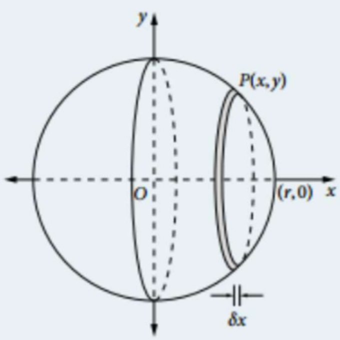
$$= \frac{1}{3}\pi r^2 h$$

Example 3

Find the volume of a sphere of radius r.

Solution

The volume of a sphere can be considered as the volume generated by rotating the semicircle defined by $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$, about the x-axis.



Hence:

$$V = \pi \int_{-r}^{r} y^2\,dx \qquad \text{where } y = \sqrt{r^2 - x^2}$$

$$= \pi \int_{-r}^{r} \left(r^2 - x^2\right)dx \qquad \text{because } y^2 = r^2 - x^2$$

$$= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right)$$

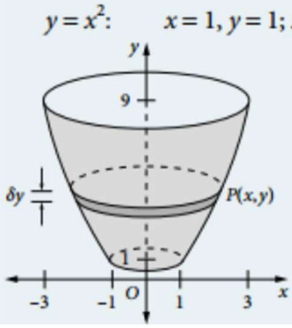
$$= \frac{4}{3}\pi r^3$$

Example 4

The part of the parabola $y = x^2$ between $x = 1$ and $x = 3$ is rotated about the y-axis. Calculate the volume generated.

Solution

$y = x^2$: $\quad x = 1, y = 1; x = 3, y = 9$



$$V = \pi \int_1^9 x^2\,dy \qquad \text{where } x^2 = y$$

$$V = \pi \int_1^9 y\,dy$$

$$= \pi\left[\frac{y^2}{2}\right]_1^9$$

$$= \pi\left(\frac{81}{2} - \frac{1}{2}\right)$$

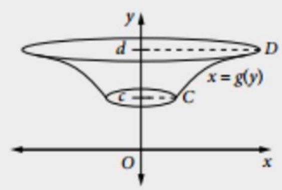
$$= 40\pi \text{ units}^3$$

Example 3, above, proves the formula for the volume of the sphere—a formula that you have used for many years. The formula for the area of a circle $A = \pi r^2$ can similarly be proved using calculus.

Rotating around the y-axis

When the arc CD of the curve $x = g(y)$ on the interval $c \le y \le d$ is rotated about the y-axis, the volume of the solid of revolution formed is given by:

$$V = \pi \int_c^d (g(y))^2 \, dy \qquad \text{or} \qquad V = \pi \int_c^d x^2 \, dy$$

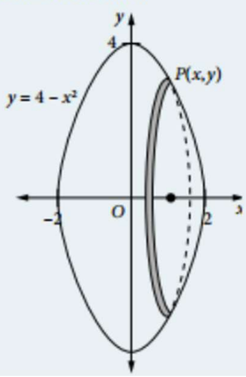


Example 5

Find the volume of the solid formed when the area bounded by the parabola $y = 4 - x^2$ and the x-axis is rotated about: **(a)** the x-axis **(b)** the y-axis.

Solution

(a) Rotate about x-axis:



$$V = \pi \int_{-2}^{2} y^2 \, dx \qquad \text{where } y = 4 - x^2$$

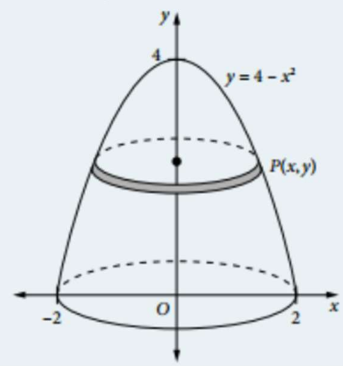
$$= \pi \int_{-2}^{2} \left(16 - 8x^2 + x^4\right) dx$$

$$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^{2}$$

$$= \pi\left(\left(32 - \tfrac{64}{3} + \tfrac{32}{5}\right) - \left(-32 + \tfrac{64}{3} - \tfrac{32}{5}\right)\right)$$

$$= \frac{512\pi}{15} \text{ units}^3$$

(b) Rotate about y-axis:



$$V = \pi \int_0^4 x^2 \, dx \qquad \text{where } x^2 = 4 - y$$

$$= \pi \int_0^4 (4 - y) dx$$

$$= \pi \left[4y - \frac{y^2}{2} \right]_0^4$$

$$= \pi((16 - 8) - 0)$$

$$= 8\pi \text{ units}^3$$

Example 6

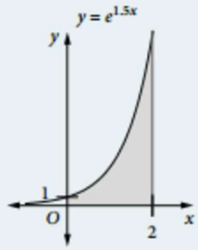
Calculate: **(a)** the area bounded by the curve $y = e^{1.5x}$, the coordinate axes and the line $x = 2$

(b) the volume obtained by rotating this area about the x-axis.

Solution

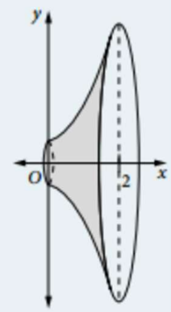
(a) $y = e^{1.5x}, y = 0, x = 2$

$$\text{Area} = \int_0^2 e^{1.5x}\,dx = \left[\frac{2}{3}e^{1.5x}\right]_0^2 = \frac{2}{3}\left(e^3 - e^0\right) = \frac{2\left(e^3-1\right)}{3} \approx 12.72 \text{ units}^2$$



(b) Volume $= \pi\int_0^2 y^2\,dx$ where $y = e^{1.5x}$.

$$= \pi\int_0^2 e^{3x}\,dx = \frac{\pi}{3}\left[e^{3x}\right]_0^2 = \frac{\pi}{3}\left(e^6 - e^0\right) = \frac{\pi\left(e^6-1\right)}{3} \approx 421.4 \text{ units}^3$$



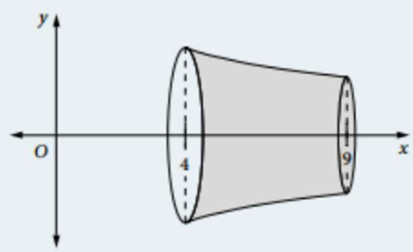
Example 8

Find the volume generated by rotating about the x-axis the area beneath the curve $y = \frac{1}{\sqrt{x}}$ between $x = 4$ and $x = 9$.

Solution

Volume $= \pi\int_4^9 y^2\,dx$ where $y = \frac{1}{\sqrt{x}}$

$$= \pi\int_4^9 \frac{1}{x}dx = \pi\left[\log_e x\right]_4^9 = \pi\left(\log_e 9 - \log_e 4\right) = \pi\log_e 2.25 \approx 2.548$$



Example 7

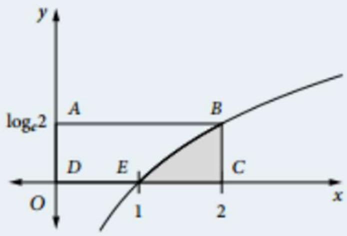
Find: **(a)** the area bounded by the curve $y = \log_e x$, the x-axis and the ordinate $x = 2$

(b) the volume of the solid of revolution formed by rotating the area bounded by the curve $y = \log_e x$, the coordinate axes and the line $y = \log_e 2$ about the y-axis.

Solution

(a) Area $= \int_1^2 \log_e x\, dx$

Instead of trying to evaluate this integral directly, draw a diagram.



This problem requires the area of the shaded region BCE. It can be obtained by finding the area of the rectangle $ABCD$ and subtracting the area $ABED$.

Because $y = \log_e x$, you can write $x = e^y$.

At $x = 2$, $y = \log_e 2$:

$$\text{Area } ABED = \int_0^{\log_e 2} e^y\, dy = \left[e^y\right]_0^{\log_e 2} = e^{\log_e 2} - e^0 = 2 - 1 = 1$$

$$\text{Area } ABCD = 2\log_e 2$$

$$\therefore \text{Area } BCE = 2\log_e 2 - 1 \approx 0.386 \text{ units}^2$$

(b) Volume $= \pi \int_0^{\log_e 2} x^2\, dy$ where $x = e^y$

$$= \pi \int_0^{\log_e 2} e^{2y}\, dy$$

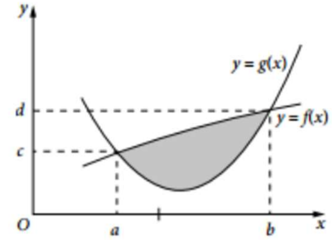
$$= \frac{\pi}{2}\left[e^{2y}\right]_0^{\log_e 2}$$

$$= \frac{\pi}{2}\left(e^{2\log_e 2} - e^0\right)$$

$$= \frac{\pi}{2}(4-1) = \frac{3\pi}{2} \text{ units}^3$$

Volume by rotating the region between two curves

When the region bounded by two curves $y = f(x)$ and $y = g(x)$ is rotated about the x-axis, the volume of the solid of revolution formed is given by $V = \pi \int_a^b \left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx$, where a and b are the abscissae of the points of intersection of the two curves, $a < b$ and $f(x) \geq g(x)$.

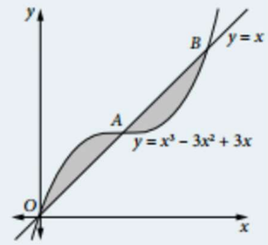


If the region is rotated about the y-axis, the equation of each curve must first be written as a function of y, i.e. $x = f{-}1(y)$ and $x = g{-}1(y)$, and the ordinates of the points of intersection used, namely c and d, as shown in the diagram.

The volume of the solid of revolution is given by $V = \pi \int_c^d \left(\left[f^{-1}(y)\right]^2 - \left[g^{-1}(y)\right]^2\right)dy$.

Example 9

The curve $y = x^3 - 3x^2 + 3x$ and the line $y = x$ intersect at (0, 0), A and B.



(a) Find the coordinates of A and B.

(b) Calculate the shaded area between the curves.

(c) The shaded region between the curves from O to A is rotated about the x-axis. Calculate the exact volume of the solid formed.

(d) The shaded region between the curves from A to B is rotated about the x-axis. Calculate the exact volume of the solid formed.

(e) Hence find the volume of the solid formed when the shaded region between the curves from O to B is rotated about the x-axis.

Solution

(a)

$$x^3 - 3x^2 + 3x = x$$
$$x^3 - 3x^2 + 2x = 0$$
$$x(x^2 - 3x + 2) = 0$$
$$x(x-1)(x-2) = 0$$
$$x = 0, 1, 2$$
$$y = 0, 1, 2$$

$A(1, 1)$ and $B(2, 2)$

(b)

$$\begin{aligned}
\text{Area} &= \int_0^1 \left(x^3 - 3x^2 + 3x - x\right)dx + \int_1^2 \left(x - \left(x^3 - 3x^2 + 3x\right)\right)dx \\
&= \int_0^1 \left(x^3 - 3x^2 + 2x\right)dx + \int_1^2 \left(-x^3 + 3x^2 - 2x\right)dx \\
&= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 + \left[-\frac{x^4}{4} + x^3 - x^2\right]_1^2 \\
&= \frac{1}{4} - 1 + 1 - 0 + \left(-4 + 8 - 4 - \left(-\frac{1}{4} + 1 - 1\right)\right) \\
&= \frac{1}{2}\ \text{unit}^2
\end{aligned}$$

(c)

$$\begin{aligned}
\text{Volume from } O \text{ to } A &= \pi \int_0^1 \left\{\left(x^3 - 3x^2 + 3x\right)^2 - x^2\right\}dx \\
&= \pi \int_0^1 \left(x^6 - 6x^5 + 15x^4 - 18x^3 + 8x^2\right)dx \\
&= \pi \left[\frac{x^7}{7} - x^6 + 3x^5 - \frac{9x^4}{2} + \frac{8x^3}{3}\right]_0^1 \\
&= \pi \left(\frac{1}{7} - 1 + 3 - \frac{9}{2} + \frac{8}{3} - 0\right) \\
&= \frac{13\pi}{42}\ \text{units}^3
\end{aligned}$$

(d)

$$\begin{aligned}
\text{Volume from } A \text{ to } B &= \pi \int_1^2 \left\{x^2 - \left(x^3 - 3x^2 + 3x\right)^2\right\}dx \\
&= -\pi \int_1^2 \left(x^6 - 6x^5 + 15x^4 - 18x^3 + 8x^2\right)dx \\
&= -\pi \left[\frac{x^7}{7} - x^6 + 3x^5 - \frac{9x^4}{2} + \frac{8x^3}{3}\right]_1^2 \\
&= -\pi \left(\frac{128}{7} - 64 + 96 - 72 + \frac{64}{3} - \left[\frac{1}{7} - 1 + 3 - \frac{9}{2} + \frac{8}{3}\right]\right) \\
&= \frac{29\pi}{42}\ \text{units}^3
\end{aligned}$$

(e) Volume from O to $B = \dfrac{13\pi}{42} + \dfrac{29\pi}{42} = \pi$ units^3