PRIMITIVE FUNCTIONS

Finding the primitive of a function

The process of finding the function from the derivative is called **anti-differentiation** or **finding the primitive** of the function.

Examples:

the derivative of x^2 is $2x$, therefore a primitive (or anti-derivative) of $2x$ is x^2

the derivative of $2\,x^4$ is $8\,x^3$, therefore a primitive (or anti-derivative) of $8\,x^3$ is $2\,x^4$

the derivative of $sin\,x$ is $cos\,x$, therefore a primitive (or anti-derivative) of $cos\,x$ is $sin\,x$

NOTE that the derivative of $x^2 + C$ (where C is a constant) is also $2x$, therefore there is not one primitive for $2x$, but a family of primitives which differ from each other by a constant, i.e. $x^2 + C$

So for example, the derivative of e^{2x} is $2\,e^{2x}$, therefore all possible primitives (or anti-derivatives) of $2\,e^{2x}$ are represented by $e^{2x} + C$ (where C is a constant).

Notation 1: normally, capital letters are used to show primitive functions, i.e. the primitive of the function $f(x)$ is generally noted $F(x)$. Therefore $F'(x) = f(x)$. Similarly, $f(x)$ is the primitive of $f'(x)$

Notation 2: the notation $\int f(x)\,dx$ is also commonly used to represent the primitive of $f(x)$. So $\int f(x)\,dx = F(x)$

Examples:

$$\int 2x\ dx = x^2 + C \qquad \int cos\,x\ dx = sin\,x + C \qquad \int sec^2 x\ dx = tan\,x + C$$

This notation is called "**the indefinite integral**". The indefinite integral gives a function to represent all possible values of the primitive by adding C, **the constant of integration.**

The term dx is used to show the variable of integration; this is necessary as some functions depend on several variables, for example: $f(x, y, z) = 2x + 3y - 4z$, so we need to show clearly the variable of integration, as $\int f(x, y, z)\ dx$ would be different from $\int f(x, y, z)\ dy$

Primitive of polynomial functions

The derivative of the function $\frac{x^{n+1}}{n+1}$ is $\frac{1}{n+1}(n+1)\,x^{n+1-1} = x^n$ therefore:

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$$

This is also valid for positive or negative, rational or not, values of n different of (-1) (the case $n = -1$ will be studied later).

Example: $$\int \frac{1}{x^3}\ dx = \int x^{-3}\ dx = \left(\frac{x^{-3+1}}{-3+1}\right) + C = \frac{x^{-2}}{(-2)} + C = -\frac{1}{2\,x^2} + C$$

Example 1

Find the equation of the curve defined by $f'(x) = 2x$ that passes through the point $(2, 8)$.

Solution

$f'(x) = 2x$, find the primitive: $f(x) = x^2 + C$

Passes through $(2, 8)$: $8 = 2^2 + C$

$C = 4$

Hence $f(x) = x^2 + 4$ is the equation of the curve.

Example 2

Find the primitive of each expression.

(a) $5x^4 + 3x^2 + 4$ **(b)** $16 - 8x + x^3$ **(c)** $x^{10} + 4$

Solution

(a) $f'(x) = 5x^4 + 3x^2 + 4$:

$f(x) = x^5 + x^3 + 4x + C$

(b) $f'(x) = 16 - 8x + x^3$:

$f(x) = 16x - \frac{8x^2}{2} + \frac{x^4}{4} + C$

$f(x) = 16x - 4x^2 + \frac{x^4}{4} + C$

(c) $f'(x) = x^{10} + 4$:

$f(x) = \frac{x^{11}}{11} + 4x + C$

Example 3

The gradient function of a curve is $3x^2 - 2x$ and the curve passes through the point $(2, 1)$. Find the equation of the curve.

Solution

$\frac{dy}{dx} = 3x^2 - 2x$, find the primitive: $y = \frac{3x^3}{3} - \frac{2x^2}{2} + C$

Simplify: $y = x^3 - x^2 + C$

At point $x = 2, y = 1$: $1 = 8 - 4 + C$

$C = -3$

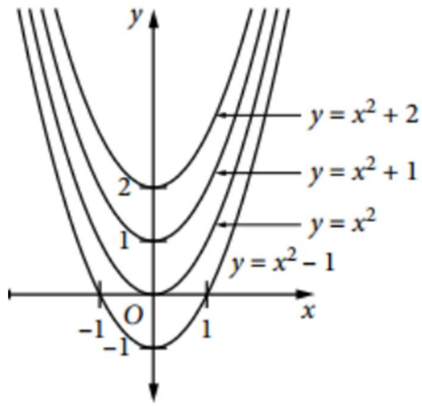
The equation of the curve is $y = x^3 - x^2 - 3$.

PRIMITIVE FUNCTIONS

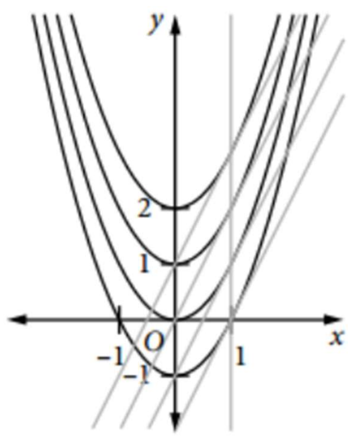
Primitives and the graph of a function

As mentioned before, $\int 2x\ dx = x^2 + C$, therefore the primitive functions of $f(x) = 2x$ are all those functions such as $F(x) = x^2 + C$ i.e. all the primitives vary only by a constant.

Graphically, they are the same curve translated vertically by a fixed amount:



If the tangent at $x = 1$ is drawn for these curves, then the tangents form a system of parallel lines, because each curve has the same gradient function.



Example 4

Find $f(x)$ given $f'(x)$.

(a) $f'(x) = x + \frac{1}{x^2}$ **(b)** $f'(x) = x\sqrt{x} + 1$ **(c)** $f'(x) = \frac{x^3 + 2x^2 + 1}{x^2}$

Solution

(a) $f'(x) = x + \frac{1}{x^2}$, rewrite using index notation: $f'(x) = x + x^{-2}$

Find primitive: $f(x) = \frac{x^2}{2} + \frac{x^{-1}}{-1} + C$

$f(x) = \frac{x^2}{2} - \frac{1}{x} + C$

(b) $f'(x) = x\sqrt{x} + 1$, rewrite using index notation: $f'(x) = x^{\frac{3}{2}} + 1$

Find primitive: $f(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + x + C$

$f(x) = \frac{2}{5}x^{\frac{5}{2}} + x + C$

(c) If there is a single term in the denominator, divide the numerator by the denominator before finding the primitive.

$f'(x) = \frac{x^3 + 2x^2 + 1}{x^2}$: $f'(x) = x + 2 + \frac{1}{x^2}$

Rewrite using indices: $f'(x) = x + 2 + x^{-2}$

Find primitive: $f(x) = \frac{x^2}{2} + 2x + \frac{x^{-1}}{-1} + C$

$f(x) = \frac{x^2}{2} + 2x - \frac{1}{x} + C$

Example 5

A colony of bats has a growth rate $\frac{dN}{dt} = 80t$, where t is the number of years since 2018 and $N(t)$ is the population. If there were 3000 bats in the year 2018, how many bats are in the colony in 2048?

Solution

$t = 0, N = 3000, \frac{dN}{dt} = 80t$

Find the primitive: $N(t) = 40t^2 + C$

But $N(0) = 3000$: $3000 = C$

$N(t) = 40t^2 + 3000$

In 2048, $t = 30$: $N(30) = 40 \times 900 + 3000 = 39\,000$

There would be 39 000 bats in the colony in 2048.