COUNTING TECHNIQUES IN PROBABILITY

1 Five cards are drawn at random from a standard pack of 52 playing cards. What is the probability that:
(a) they are all from the same suit (b) they include four Aces (c) they include three 10s and two 9s?

$\frac{4 \times {}^{13}C_5}{{}^{52}C_5} = \frac{33}{16,660}$ $\frac{48}{{}^{52}C_5} = \frac{1}{54,145}$ $\frac{{}^4C_3 \times {}^4C_2}{{}^{52}C_5} = \frac{1}{108,290}$

5R 4W 3 Balls

2 A bag contains five red balls and four white balls. Three balls are withdrawn without replacement. The probability of drawing at least two red balls is:
A $\frac{1}{21}$ B $\frac{5}{42}$ C $\frac{20}{42}$ $\boxed{D}$ $\frac{25}{42}$

3 Reds: $\frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} = \frac{5}{42}$

4 Reds: $\frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} + \frac{4}{9} \times \frac{5}{8} \times \frac{4}{7} + \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = 3 \times \frac{50 \times 4^2}{504} = \frac{20}{42}$

TOTAL 25/42

3 A student writes a random three-digit number using the digits 1–9. What is the probability that the three digits in the number are all the same?

$9 \times \frac{1}{9} \times \frac{1}{9} \times \frac{1}{9} = \frac{1}{9^2} = \frac{1}{81}$

3W 4R 5B 3 Balls

5 A bag contains three white balls, four red balls and five black balls. Three balls are drawn at random without replacement. What is the probability that they are: (a) different colours (b) the same colour?

a) $\frac{3}{12} \times \frac{4}{11} \times \frac{5}{10} \times 6 = \frac{3}{11}$

b) $\frac{3 \times 2 \times 1}{12 \times 11 \times 10} + \frac{4 \times 3 \times 2}{12 \times 11 \times 10} + \frac{5 \times 4 \times 3}{12 \times 11 \times 10} = \frac{6 + 24 + 60}{1320} = \frac{3}{44}$

7T 5S 7 people

6 From a group of seven teachers and five students, a random selection of seven people is made. What is the probability that the selection contains at least four teachers?

4 teachers: ${}^7C_4 \times {}^5C_3 = 350$
5 teachers + 2 students : ${}^7C_5 \times {}^5C_2 = 210$
6 teachers + 1 student: ${}^7C_6 \times {}^5C_1 = 35$
7 teachers + no students : ${}^7C_7 \times {}^5C_0 = 1$

TOTAL 596

Total number of possibilities : ${}^{12}C_7 = 792$

so $p = \frac{596}{792} = \frac{149}{198}$

7 A committee of three judges and four lawyers is to be chosen from six judges and seven lawyers. What is the probability that it contains a particular judge and a particular lawyer?

Total number of combinations: ${}^6C_3 \times {}^7C_4 = 700$

If we remove 1 judge and 1 lawyer, that's ${}^5C_2 \times {}^6C_3 = 200$

So $p = \frac{200}{700} = 2/7$

8 Five cards are drawn from a standard pack of 52 playing cards (without replacement). What is the probability of drawing at least three Aces?

$$\left.\begin{array}{l} \text{4 aces AND 1 other card} = {}^4C_4 \times {}^{48}C_1 = 48 \\ \text{3 aces AND 2 other cards} = {}^4C_3 \times {}^{48}C_2 = 4512 \end{array}\right\} \text{TOTAL } 4{,}560$$

So $p = \frac{4560}{{}^{12}C_5} = \frac{19}{10{,}829}$

9 A person correctly picks the first and second horses in a race of 10 horses. If each horse was equally likely to win, what is the probability of this? The order is important, so ${}^{10}P_2 = 90$

Only 1 possibility : $\frac{1}{90}$

10 The letters of the word PROMISE are arranged in a row. What is the probability that there are three letters between 'P' and 'R'? There's no repeat. P could be in the 1st, 2nd or 3rd position, so 3 possibilities. Then the position of R is known. Then 5! for the remaining 3 letters, so $3 \times 5! = 360$. Same number if R is in the 1st, 2nd or 3rd position. Then the total number of possible arrangements is 7!. So $p = \frac{2 \times 360}{7!} = \frac{1}{7}$

11 Six giraffes randomly arrange themselves in a line. What is the probability that the tallest giraffe and the shortest giraffe are together?

The tallest giraffe can be in position 1 to 5 (5 possibilities) and then the position of the shortest giraffe is known, and then 4! for the last 4.

Same if the shortest is in position 1 to 5 and the tallest goes after; so $5 \times 4!$ too.

Then the total number of possible arrangements is 6!

So $p = \frac{2 \times 5 \times 4!}{6!} = \frac{1}{3}$

COUNTING TECHNIQUES IN PROBABILITY

4E, 3N, 2D

14 The letters of the word INDEPENDENCE are arranged in a row. What is the probability of all the letters 'E' being together? Indicate whether each statement below is a correct or incorrect step in solving this problem.

(a) Number of arrangements of the letters = $\frac{12!}{4!3!2!}$ correct

(b) Number of ways that the letters 'E' are together = $\frac{9!}{4!3!2!}$ incorrect

(c) Number of ways that the letters 'E' are together = $\frac{9!}{3!2!}$ correct

(d) Probability of the letters 'E' being together = $\frac{1}{55}$ correct

For c) $\frac{9 \times 8!}{3! \times 2!} = \frac{9!}{3!\,2!} = 30,240$

$P(E_s \text{ being together}) = \frac{9!/3!2!}{12!/4!3!2!} = \frac{9!\,4!}{12!} = \frac{1}{55}$

15 A carton contains a dozen eggs, of which three have a double yolk. If three eggs are taken to make a cake, find the probability that the three eggs all have double yolks.

$\frac{3}{12} \times \frac{2}{11} \times \frac{1}{10} = \frac{1}{220}$

16 A team of six is selected from 10 people. What is the probability that the youngest and oldest people are on the team?

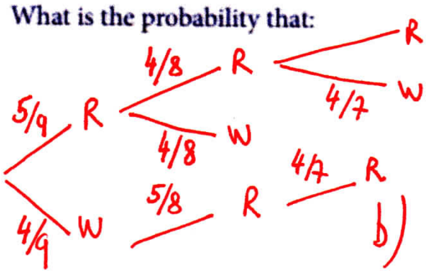
$P = \frac{{}^{8}C_4}{{}^{10}C_6} = \frac{70}{210} = \frac{1}{3}$

17 A box contains five red cubes and four white cubes. Three cubes are drawn in succession without replacement. What is the probability that:

5R 4W

(a) the first two cubes are red and the third cube is white

(b) any two cubes are red and one cube is white?



a) $\frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = \frac{10}{63}$

b) $\frac{10}{63} + \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} + \frac{4}{9} \times \frac{5}{8} \times \frac{4}{7} = \frac{30}{63} = \frac{10}{21}$

18 An angler has caught 15 fish, of which three are undersized. A random sample of three fish is drawn without replacement by an inspector. The angler is fined if one or more of the fish in the sample is undersized. What is the probability that the angler is fined?

P(at least one undersized) = 1 − P(none undersized)

$$= 1 - \frac{12}{15} \times \frac{11}{14} \times \frac{10}{13}$$

$$= 1 - \frac{44}{91} = \frac{47}{91}$$

19 Eight different jackets are divided into two piles. What is the probability that there will be four in each pile?

There could be 1 jacket in 1 pile ($^8C_1$), or 2 jackets in one pile ($^8C_2$), or 3 jackets in 1 pile ($^8C_3$), or 4 in 1 pile ($^8C_4$).

So probability of 4 jackets in 1 pile $= \frac{^8C_4}{^8C_1 + {^8C_2} + {^8C_3} + {^8C_4}} = \frac{35}{81}$

20 A party of 12 people, including *A* and *B*, are arranged at random in a straight line. What is the probability that *A* and *B* are not next to one another?

P(A and B not next to one another) = 1 − P(A and B next to one another)

$$= 1 - \frac{[11 \times 2] \times 10!}{12!}$$

as there are 11 choices for the 1st letter (either A or B). then 10! for the remaining people

$$= 1 - \frac{2}{12} = 1 - \frac{1}{6} = \frac{5}{6}$$

21 A box contains 15 memory cards, of which 5 are defective. If a random sample of 6 memory cards is drawn from the box (without replacement), determine the probability of there being 0, 1, 2, 3, 4, 5 defective memory cards in the sample.

$P(0 \text{ defective}) = \frac{^{10}C_6}{^{15}C_6} = \frac{6}{143} \approx 0.4196$

$P(1 \text{ defective}) = \frac{5 \times {^{10}C_5}}{^{15}C_6} = \frac{36}{143} \approx 0.2517$

$P(2 \text{ defective}) = \frac{^5C_2 \times {^{10}C_4}}{^{15}C_6} = \frac{2100}{5005} \approx 0.4196$

$P(3 \text{ defective}) = \frac{^5C_3 \times {^{10}C_3}}{^{15}C_6} = \frac{1200}{5005} \approx 0.2398$

$P(4 \text{ defective}) = \frac{^5C_4 \times {^{10}C_2}}{^{15}C_6} = \frac{225}{5005} \approx 0.0450$

$P(5 \text{ defective}) = \frac{^5C_5 \times {^{10}C_1}}{^{15}C_6} = \frac{10}{5005} \approx 0.0020$

COUNTING TECHNIQUES IN PROBABILITY

5R 7B

22 An urn contains 12 distinguishable cubes of which five are red and the remainder black. If a random sample of six cubes is drawn without replacement, calculate the probabilities of 0, 1, 2, 3, 4, 5 red cubes in the sample.

$$P(0R) = \frac{{}^5C_0\ {}^7C_6}{{}^{12}C_6} = \frac{7}{924} = \frac{1}{132} \qquad P(1R) = \frac{{}^5C_1 \times {}^7C_5}{{}^{12}C_6} = \frac{105}{924} = \frac{5}{44}$$

$$P(2R) = \frac{{}^5C_2 \times {}^7C_4}{{}^{12}C_6} = \frac{350}{924} = \frac{25}{66} \qquad P(3R) = \frac{{}^5C_3 \times {}^7C_3}{{}^{12}C_6} = \frac{350}{924} = \frac{25}{66}$$

$$P(4R) = \frac{{}^5C_4 \times {}^7C_2}{{}^{12}C_6} = \frac{105}{924} = \frac{5}{44} \qquad P(5R) = \frac{{}^5C_5 \times {}^7C_1}{{}^{12}C_6} = \frac{7}{924} = \frac{1}{132}$$

3B 5W

23 Two boxes each contain eight balls. In box *A* there are three black and five white balls; in box *B* there are one black and seven white balls. For each box, find the probability that two balls chosen at random without replacement will both be white.

1B 7W

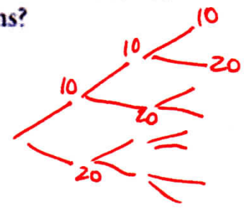
$$P(2W \mid \text{Box} A) = \frac{5}{8} \times \frac{4}{7} = \frac{20}{56} = \frac{5}{14}$$

$$P(2W \mid \text{Box} B) = \frac{7}{8} \times \frac{6}{7} = \frac{42}{56} = \frac{3}{4}$$

3 C 8C 4 x 10c

24 A sample of three coins is selected without replacement from a handful of eight coins that consists of four 10c coins and four 20c coins. What is the probability that the sample contains at least two 10c coins?

4 x 20c



It could contains 2 10cents or 3 10cents coins.

$$P = \frac{4}{8} \times \frac{3}{7} \times \frac{2}{6} + \frac{4}{8} \times \frac{3}{7} \times \frac{4}{6} + \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} + \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6}$$

$$P = \frac{24}{336} + 3 \times \frac{4 \times 3}{336} = \frac{1}{14} + \frac{3}{7} = \frac{1}{2}$$

25 A hand of five cards is dealt from a standard pack of 52 playing cards. What is the probability that it contains at least one Ace?

$$P(\text{at least one Ace}) = 1 - P(\text{no Ace})$$

$$= 1 - \frac{{}^{48}C_5}{{}^{52}C_5} = 1 - \frac{1712,304}{2,598,960}$$

$$\simeq 0.3412$$

COUNTING TECHNIQUES IN PROBABILITY

8P 4C 4

26 From a group of 12 people, of whom eight are painters and four are carpenters, a sample of four is selected at random. What is the probability that the sample contains at least two carpenters?

P(at least 2 carpenters) = 1 − [P(no carpenters) + P(1 carpenter)]

$$= 1 - \left[\frac{{}^{8}C_4}{{}^{12}C_4} + \frac{{}^{4}C_1 \times {}^{8}C_3}{{}^{12}C_4}\right] = 1 - \frac{70}{495} - \frac{224}{495}$$

$$= \frac{67}{165}$$

6W 4B 2W 2B

27 Box *A* contains six white and four black balls. Box *B* contains two white and two black balls. From box *A*, two balls are selected at random and placed in box *B*. Two balls are then selected at random from box *B*. What is the probability that exactly one of these two balls is white?

P(1W) = P(1W given 0W picked from A) + P(1 white given 1W picked from A) + P(1 white | 2W picked from A)

$$P(1W) = \left[\frac{4}{10}\times\frac{3}{9}\right]\times\left[\frac{2}{6}\times\frac{4}{5}+\frac{4}{6}\times\frac{2}{5}\right] + \left[\frac{6}{10}\times\frac{4}{9}+\frac{4}{10}\times\frac{6}{9}\right]\times\left[\frac{3}{6}\times\frac{3}{5}+\frac{3}{6}\times\frac{3}{5}\right] + \left[\frac{6}{10}\times\frac{5}{9}\right]\times\left[\frac{4}{6}\times\frac{2}{5}+\frac{2}{6}\times\frac{4}{5}\right]$$

$$P(1W) = \frac{16}{225} + \frac{8}{25} + \frac{8}{45} = \frac{128}{225}$$

28 From a set of 10 cards numbered 1–10, two cards are drawn without replacement. What is the probability that:

(a) both numbers are even

(b) one is even and the other is odd

(c) the sum of the two numbers is 12

(d) both numbers are even and the sum of the two numbers is 12?

a) $\frac{1}{2} \times \frac{4}{9} = \frac{2}{9}$

b) $\frac{1}{2} \times \frac{5}{9} + \frac{1}{2} \times \frac{5}{9} = \frac{5}{9}$ (even 1st, then odd or vice versa)

c) could be 10+2, 9+3, 8+4, 7+5, $\cancel{6+6}$ NO, 5+7, 4+8, 3+9, 2+10

So 8 possibilities. so $\frac{4}{{}^{10}C_2} = \frac{4}{45}$

d) could be 10+2 or 8+4

so $\frac{2}{{}^{10}C_2} = \frac{2}{45}$

36 A scientific study uses the 'capture–recapture' technique. In the first stage of the study, 48 possums are caught, tagged and then released. Later, in the second stage of the study, some possums are again captured from the same area. Of these possums, 24 of them are found to be tagged, which is 40% of the total captured in this second stage.

(a) In the second stage of the study, how many possums are captured in total?

(b) Calculate the estimate for the total population of possums in this area.

a) 24 possums is 40% so the total number of possums captured is $\frac{24}{0.4} = 60$.

b) Assuming that the numbers of possums tagged is, the same in the sample of 60 possums, and in the general population, that means that 40% of the total population is tagged. So there must be a total of $\frac{48}{0.4} = 120$ possums.