SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = g(y)$

1 In each case, find the equation of the solution curve and then sketch its graph.

(a) $\frac{dy}{dx} = -y,\ y(0) = 1$ **(b)** $\frac{dy}{dx} = 2y,\ y(0) = -1$

a) $\frac{dy}{dx} = -y \iff \frac{dy}{y} = -dx$

Now that the variables have been separated, we can integrate both sides separately.

$\int \frac{dy}{y} = \int -dx$ so $\ln|y| = -x + C$ so $y = Ae^{-x}$ (general solution)

But $y(0) = 1$ so $Ae^{-0} = 1$ or $A = 1$

$\therefore y = e^{-x}$ is the particular solution for which $y(0) = 1$

b) $\frac{dy}{dx} = 2y \iff \frac{dy}{y} = 2\,dx$

$\Rightarrow \int \frac{dy}{y} = \int 2\,dx \Rightarrow \ln|y| = 2x + C$

So $y = Ae^{2x}$ is the general solution

But $y(0) = -1$ so $Ae^{2\times 0} = A = -1$

So $y = -e^{2x}$ is the particular solution of the differential equation for which $y(0) = -1$

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = g(y)$

2 In each case, find the equation of the solution curve and then sketch its graph.

(a) $\frac{dy}{dx} = -2(y-3)$, $y(0) = 8$ **(b)** $\frac{dy}{dx} = -2(y(x)-8)$, $y(0) = 3$

a) $\Leftrightarrow \frac{dy}{y-3} = -2\,dx$ so $\int \frac{dy}{y-3} = \int -2\,dx$

$\ln|y-3| = -2x + C$ so $y-3 = Ae^{-2x}$

or $y = 3 + Ae^{-2x}$ which is the general solution.

But $y(0) = 8$ so $3 + Ae^{-2\times 0} = 3 + A = 8$ so $A = 5$

$y = 3 + 5e^{-2x}$ is the particular solution for which $y(0) = 8$

b) $\frac{dy}{dx} = -2(y-8) \Leftrightarrow \frac{dy}{y-8} = -2\,dx$

$\int \frac{dy}{y-8} = \int -2\,dx$ so $\ln|y-8| = -2x + C$

$y - 8 = Ae^{-2x}$ is the general solution.

$\Leftrightarrow y = 8 + Ae^{-2x}$

But $y(0) = 3$ or $8 + Ae^{-2\times 0} = A + 8 = 3$

so $A = -5$

So $y = 8 - 5e^{-2x}$ is the particular solution for which $y(0) = 3$

$\frac{dy}{\cos^2 y} = dx$ so $\int \frac{dy}{\cos^2 y} = \int dx$ or $\int \sec^2 y\,dy = x + C$

$\tan y = x + C$ general solution

$y = \tan^{-1}(x + C)$

3 Given that $\frac{dy}{dx} = \cos^2 y$ and that $y = \frac{\pi}{4}$ at $x = 0$, then which of the following is true?

A $y = \frac{1}{2}y + \frac{1}{4}\sin 2y$ **B** $x = \tan\left(y + \frac{\pi}{4}\right)$ $\boxed{\textbf{C}}$ $y = \tan^{-1}(x+1)$ **D** $y = \tan^{-1}\left(x - \frac{\pi}{4}\right)$

When $x = 0$ $y(0) = \tan^{-1}(0 + C) = \tan^{-1}(C) = \pi/4$ so $C = 1$

$y = \tan^{-1}(x + 1)$

4 In Biology and Ecology, the term *desiccation* refers to the drying out (i.e. the loss of water) of the cells of a living organism. Most cells are mostly made of water. Assume that the desiccation of a cell is modelled by the solution of the following differential equation: $\frac{dV}{dt} = -kV^{\frac{2}{3}}$, where V is the volume of the water in the cell, t is time and k is an appropriate constant of proportionality.

During the intense heat of the Australian summer, the cells of a newly fallen eucalyptus leaf still contain water, but the leaf loses this water rapidly through the process of desiccation. Suppose that each leaf cell initially contains 8 μm^3 of water, but 4 hours later each cell has only 1 μm^3 of water.

(a) Find the particular solution of $\frac{dV}{dt} = -kV^{\frac{2}{3}}$, $V(0) = 8\,\mu m^3$.

(b) Find the time taken for the cells to lose all their water (assuming that the environmental conditions don't change over this time).

a) $\frac{dV}{dt} = -kV^{2/3} \iff \frac{dV}{V^{2/3}} = -k\,dt$

So $\int \frac{dV}{V^{2/3}} = \int -k\,dt \iff \frac{V^{-2/3+1}}{\left(-\frac{2}{3}+1\right)} = -kt + C$

$\iff 3V^{1/3} = -kt + C$ so $V^{1/3} = \frac{1}{3}(-kt + C)$. general solution

But $V(0) = 8$, so $\frac{1}{3}(-k \times 0 + C) = 8^{1/3}$ so $\frac{C}{3} = 2$ $C = 6$

$V^{1/3} = \frac{1}{3}(-kt + 6) = -\frac{kt}{3} + 2.$

At $t = 4$, $V = 1$, so $1^{1/3} = \frac{1}{3}(-k \times 4 + 6)$

so $-4k + 6 = 3$ so $-4k = -3$ so $k = \frac{3}{4}$

So $V^{1/3} = -\frac{t}{4} + 2$ or $V(t) = \left(2 - \frac{t}{4}\right)^3$

b) $V(t) = 0$ when $2 - \frac{t}{4} = 0$, i.e. when $t = 8$

5 The pressure of the atmosphere, P kilopascals (kPa), decreases according to the height h km above sea level. The rate of change of the pressure with respect to the height above sea level is proportional to the pressure at that height.

(a) Write a differential equation to describe this situation.

(b) The pressure at sea level is 101.3 kPa and it is approximately 37.3 kPa at a height of 5 km. Solve the differential equation to find P as a function of h.

(c) Estimate the air pressure at the top of Mount Everest, which is about 9 km high.

a) $\frac{dP}{dh} = kP$

b) $\Leftrightarrow \quad \frac{dP}{P} = k\,dh \quad$ so $\int \frac{dP}{P} = \int k\,dh$

so $\ln|P| = kh + C \quad$ or $\quad P = Ae^{kh}$ general solution of the d.e.

When $h = 0$, $P = 101.3 \quad$ so $101.3 = Ae^{k \times 0} = A$

$\therefore A = 101.3$

$P(h) = 101.3 \times e^{kh}$

When $h = 5$, $P = 37.3 \quad$ so $37.3 = 101.3 \times e^{5k}$

so $e^{5k} = \frac{37.3}{101.3} = \frac{373}{1013} \quad$ so $5k = \ln\left[\frac{373}{1013}\right]$

$k = \frac{1}{5}\ln\left[\frac{373}{1013}\right] \simeq -0.2000$

c) $P(h) = 101.3 \times e^{-0.2 \times h}$

When $h = 9 \qquad P(9) = 101.3 \times e^{-0.2 \times 9}$

$P(9) = 16.7$ kPa

6 In an electric circuit, a capacitor of capacitance C charged to a potential difference E is discharged through a resistance R. If q is the charge on the capacitor at time t, then $\frac{dq}{dt} = -\frac{q}{RC}$ is the differential equation describing this situation. If initially $q = EC$ then find the solution of this equation (that is, q as a function of t).

$\frac{dq}{dt} = -\frac{q}{RC}$ so $\frac{dq}{q} = -\frac{dt}{RC}$

$\int \frac{dq}{q} = \int -\frac{dt}{RC} = -\frac{1}{RC}\int dt$ so $\ln|q| = -\frac{t}{RC} + K$

$q = A e^{-t/RC}$ general solution of the d.e.

At $t = 0$ $q = EC$ so: $EC = A e^{-0/RC} = A$

so $A = EC$

$q = EC e^{-t/RC}$

7 Newton's law of cooling states that 'the cooling rate of a body is proportional to the difference between the temperature of the body and that of the surrounding medium.' This may be written as $\frac{dT}{dt} = -k(T - M)$, where T is the temperature at any time t and M is the temperature of the surrounding medium (a constant).

A pot of soup is cooked at 100°C. To cool the soup, it is placed in a room where the temperature is 20°C. After 20 minutes the temperature of the soup has dropped to 70°C.

(a) Find the general solution of the differential equation $\frac{dT}{dt} = -k(T - 20)$.
(b) Find the value of k.
(c) How much time will it take the pot of soup to cool to 25°C?

a) $\frac{dT}{dt} = -k(T-M)$ so $\frac{dT}{T-M} = -k\,dt$

$\int \frac{dT}{T-M} = \int -k\,dt = -k\int dt$ so $\ln|T-M| = -kt + C$

so $T - M = Ae^{-kt}$ or $T(t) = M + Ae^{-kt}$

general solution of the d.e.

b) $M = 20°C$ (temperature of the room)

When $t = 0$, $T(0) = 100°C$

So $100 = 20 + Ae^{-k \times 0} = 20 + A$ so $A = 80$

$T(t) = 20 + 80e^{-kt}$

When $t = 20$, $T(20) = 70$

so $70 = 20 + 80e^{-k \times 20}$ so $e^{-20k} = \frac{70-20}{80} = \frac{5}{8} = 0.625$

so $-20k = \ln\left(\frac{5}{8}\right)$ $k = \frac{1}{20}\ln(8/5) \simeq -0.023500$

c) $T(t) = 25°C$ when $20 + 80e^{-kt} = 25$, i.e. when

$e^{-kt} = \frac{25-20}{80} = \frac{5}{80} = \frac{1}{16}$ so $-kt = \ln 1/16$

$t = -\frac{1}{k}\ln\left(\frac{1}{16}\right) = \frac{\ln 16}{k} = \frac{20\ln(16)}{\ln(8/5)} \simeq 118$ minutes

so approx 2 hours.