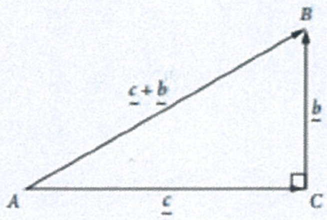
USING VECTORS IN GEOMETRIC PROOFS

The scalar product is used in geometrical proofs involving parallel and perpendicular lines, and squares on the sides of figures.

Example 13

Prove the theorem of Pythagoras in the right-angled triangle ABC, given in the diagram.



Solution

You have to prove that $|\overrightarrow{AB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{CB}|^2$.

Let $\overrightarrow{AC} = \underset{\sim}{c}$ and $\overrightarrow{CB} = \underset{\sim}{b}$.

Hence $\overrightarrow{AB} = \underset{\sim}{c} + \underset{\sim}{b}$

Since $\overrightarrow{AC} \perp \overrightarrow{CB}$, then $\underset{\sim}{c} \bullet \underset{\sim}{b} = 0$

Now $(\underset{\sim}{c} + \underset{\sim}{b}) \bullet (\underset{\sim}{c} + \underset{\sim}{b}) = \underset{\sim}{c} \bullet \underset{\sim}{c} + 2\underset{\sim}{c} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{b}$

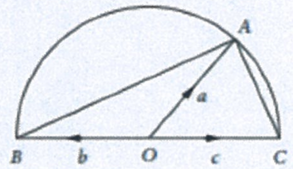
$$|\underset{\sim}{c} + \underset{\sim}{b}|^2 = |\underset{\sim}{c}|^2 + 0 + |\underset{\sim}{b}|^2$$

$$|\overrightarrow{AB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{CB}|^2$$

Hence the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Example 14

Prove that an angle inscribed in a semicircle is a right angle.



Solution

The diagram shows the semicircle ACB with centre O and diameter BOC.
A is a point on the circumference.

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$, where $|\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{c}|$ as they are all radii.

Now $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$$= -\underset{\sim}{a} + \underset{\sim}{b}$$

$$= \underset{\sim}{b} - \underset{\sim}{a}$$

And $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$$= -\underset{\sim}{a} + \underset{\sim}{c}$$

$$= \underset{\sim}{c} - \underset{\sim}{a}$$

Thus $\overrightarrow{AB} \bullet \overrightarrow{AC} = (\underset{\sim}{b} - \underset{\sim}{a}) \bullet (\underset{\sim}{c} - \underset{\sim}{a})$

$$= \underset{\sim}{b} \bullet \underset{\sim}{c} - \underset{\sim}{b} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet \underset{\sim}{c} + \underset{\sim}{a} \bullet \underset{\sim}{a}$$

But $\underset{\sim}{c} = -\underset{\sim}{b}$

So $\overrightarrow{AB} \bullet \overrightarrow{AC} = \underset{\sim}{b} \bullet (-\underset{\sim}{b}) - \underset{\sim}{b} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet (-\underset{\sim}{b}) + \underset{\sim}{a} \bullet \underset{\sim}{a}$

$$= -\underset{\sim}{b} \bullet \underset{\sim}{b} - \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{a}$$

$$= |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$$

$$= 0 \text{ since } |\underset{\sim}{a}| = |\underset{\sim}{b}|$$

Hence $\overrightarrow{AC} \perp \overrightarrow{AB}$ so $\angle BAC = 90°$

Example 15

The position vectors of the points P, Q, R and S are respectively $4\underline{i}+3\underline{j}-\underline{k}$, $5\underline{i}+2\underline{j}+2\underline{k}$, $2\underline{i}-2\underline{j}-3\underline{k}$ and $4\underline{i}-4\underline{j}+3\underline{k}$.

(a) Show that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{RS}$.

(b) Is $PQRS$ a parallelogram?

Solution

(a)

$$\overrightarrow{PQ} = 5\underline{i}+2\underline{j}+2\underline{k}-\left(4\underline{i}+3\underline{j}-\underline{k}\right)$$
$$= \underline{i}-\underline{j}+3\underline{k}$$
$$\overrightarrow{RS} = 4\underline{i}-4\underline{j}+3\underline{k}-\left(2\underline{i}-2\underline{j}-3\underline{k}\right)$$
$$= 2\underline{i}-2\underline{j}+6\underline{k}$$
$$= 2\left(\underline{i}-\underline{j}+3\underline{k}\right)$$

Since $\overrightarrow{RS} = 2\overrightarrow{PQ}$, then $\overrightarrow{PQ}$ is parallel to $\overrightarrow{RS}$.

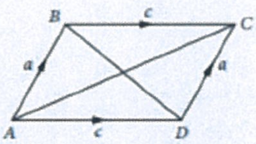
(b)

$$\left|\overrightarrow{PQ}\right| = \sqrt{1^2+1^2+3^2} = \sqrt{11}$$
$$\left|\overrightarrow{RS}\right| = \sqrt{2^2+2^2+6^2} = 2\sqrt{11}$$

Since $\left|\overrightarrow{PQ}\right| \uparrow \left|\overrightarrow{RS}\right|$, then $PQRS$ is not a parallelogram, it is a trapezium.

Example 16

Prove that the sum of the squares of the lengths of the diagonals of any parallelogram is equal to the sum of the squares of the lengths of the sides.



Solution

It is required to show that:
$$\left|\overrightarrow{BD}\right|^2+\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{CD}\right|^2+\left|\overrightarrow{DA}\right|^2$$
$$= 2\left(\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{BC}\right|^2\right)$$

Denote $\overrightarrow{AB}$ and $\overrightarrow{DC}$ by $\underline{a}$, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ by $\underline{c}$, as shown in the diagram.

From BCD: $\overrightarrow{BD} = \overrightarrow{BC}+\overrightarrow{CD} = \underline{c}-\underline{a}$

From ACD: $\overrightarrow{AC} = \overrightarrow{AD}+\overrightarrow{DC} = \underline{c}+\underline{a}$

$$\left|\overrightarrow{BD}\right|^2+\left|\overrightarrow{AC}\right|^2 = \overrightarrow{BD}\bullet\overrightarrow{BD}+\overrightarrow{AC}\bullet\overrightarrow{AC}$$
$$= \left(\underline{c}-\underline{a}\right)\bullet\left(\underline{c}-\underline{a}\right)+\left(\underline{c}+\underline{a}\right)\bullet\left(\underline{c}+\underline{a}\right)$$
$$= \underline{c}\bullet\underline{c}-2\underline{c}\bullet\underline{a}+\underline{a}\bullet\underline{a}+\underline{c}\bullet\underline{c}+2\underline{c}\bullet\underline{a}+\underline{a}\bullet\underline{a}$$
$$= 2\left(\underline{c}\bullet\underline{c}+\underline{a}\bullet\underline{a}\right)$$
$$= 2\left(c^2+a^2\right)$$
$$= 2\left(\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{BC}\right|^2\right)$$

which is the required result.