THE QUOTIENT RULE

1 Use the quotient rule to differentiate each function.

(a) $y = \dfrac{x-1}{x+1}$ (b) $f(x) = \dfrac{3x-7}{4x+5}$ (c) $g(t) = \dfrac{2t+5}{t+2}$

a) $y = \dfrac{u(x)}{v(x)}$ so $\dfrac{dy}{dx} = \dfrac{u'v - v'u}{v^2}$

$u(x) = x - 1$ so $u'(x) = 1$

$v(x) = x + 1$ $v'(x) = 1$

$\dfrac{dy}{dx} = \dfrac{1 \times (x+1) - 1 \times (x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$

b) $u(x) = 3x - 7$ $u'(x) = 3$

$v(x) = 4x + 5$ $v'(x) = 4$

$f'(x) = \dfrac{3 \times (4x+5) - 4 \times (3x-7)}{(4x+5)^2}$

$f'(x) = \dfrac{12x + 15 - 12x + 28}{(4x+5)^2} = \dfrac{43}{(4x+5)^2}$

c) $u(t) = 2t + 5$ $u'(t) = 2$

$v(t) = t + 2$ $v'(t) = 1$

$g'(t) = \dfrac{2(t+2) - 1 \times (2t+5)}{(t+2)^2}$

$g'(t) = \dfrac{2t + 4 - 2t - 5}{(t+2)^2}$

$g'(t) = -\dfrac{1}{(t+2)^2}$

THE QUOTIENT RULE

1 Use the quotient rule to differentiate each function.

(g) $y = \dfrac{4x^2}{2x+5}$ **(h)** $y = \dfrac{4x^2-2}{x^2+5}$ **(i)** $v(x) = \dfrac{x+1}{x^3-1}$

g) $u(x) = 4x^2$ $u'(x) = 8x$

$v(x) = 2x+5$ $v'(x) = 2$

$$\frac{dy}{dx} = \frac{8x(2x+5) - 2(4x^2)}{(2x+5)^2} = \frac{8x^2+40x}{(2x+5)^2} = \frac{8x(x+5)}{(2x+5)^2}$$

h) $u(x) = 4x^2-2$ $u'(x) = 8x$

$v(x) = x^2+5$ $v'(x) = 2x$

$$\frac{dy}{dx} = \frac{8x(x^2+5) - 2x(4x^2-2)}{(x^2+5)^2}$$

$$\frac{dy}{dx} = \frac{8x^3+40x-8x^3+4x}{(x^2+5)^2} = \frac{44x}{(x^2+5)^2}$$

i) $u(x) = x+1$ $u'(x) = 1$

$W(x) = x^3-1$ $W'(x) = 3x^2$

$$v'(x) = \frac{1 \times (x^3-1) - 3x^2(x+1)}{(x^3-1)^2}$$

$$v'(x) = \frac{x^3-1-3x^3-3x^2}{(x^3-1)^2}$$

$$v'(x) = \frac{-2x^3-3x^2-1}{(x^3-1)^2} = -\frac{2x^3+3x^2+1}{(x^3-1)^2}$$

THE QUOTIENT RULE

3 Differentiate each function with respect to x.

(a) $y = \dfrac{\sqrt{x+1}}{x}$ **(b)** $f(x) = \dfrac{(x+1)^2}{x}$ **(c)** $y = \dfrac{x}{(x+1)^2}$

a) $u(x) = \sqrt{x+1}$ $\quad u'(x) = \dfrac{1}{2\sqrt{x+1}}$

$v(x) = x$ $\quad v'(x) = 1$

$$\frac{dy}{dx} = \frac{\dfrac{x}{2\sqrt{x+1}} - 1 \times \sqrt{x+1}}{x^2} = \frac{x - 2\left(\sqrt{x+1}\right)^2}{2x^2\sqrt{x+1}} = \frac{x - 2(x+1)}{2x^2\sqrt{x+1}} = \frac{-(x+2)}{2x^2\sqrt{x+1}}$$

b) $$f(x) = \frac{x^2+2x+1}{x} = x + 2 + \frac{1}{x} = x + 2 + x^{-1}$$

$$f'(x) = 1 + (-1)x^{-1-1} = 1 - \frac{1}{x^2}$$

(which is faster than using the quotient rule)

c) $u(x) = x$ $\quad u'(x) = 1$

$v(x) = x^2+2x+1$ $\quad v'(x) = 2x+2$

$$\frac{dy}{dx} = \frac{1 \times (x^2+2x+1) - (2x+2) \times x}{(x+1)^4}$$

$$\frac{dy}{dx} = \frac{x^2+2x+1-2x^2-2x}{(x+1)^4}$$

$$\frac{dy}{dx} = \frac{-x^2+1}{(x+1)^4} = -\frac{(x^2-1)}{(x+1)^4} = -\frac{(x+1)(x-1)}{(x+1)^4}$$

So $$\frac{dy}{dx} = -\frac{(x-1)}{(x+1)^3} = \frac{1-x}{(x+1)^3}$$

THE QUOTIENT RULE

3 Differentiate each function with respect to x.

(d) $y=\dfrac{(2x+1)^3}{(3-x^2)^2}$ **(e)** $f(x)=\dfrac{x}{\sqrt{x+1}}$ **(f)** $y=\left(\dfrac{x+1}{x}\right)^2$

d) $y=\dfrac{8x^3+12x^2+6x+1}{9-6x^2+x^4}$ $u'(x)=24x^2+24x+6$

$v'(x)=4x^3-12x$

$$\frac{dy}{dx}=\frac{(24x^2+24x+6)(9-6x^2+x^4)-(4x^3-12x)(8x^3+12x^2+6x+1)}{(3-x^2)^4}$$

$$\frac{dy}{dx}=\frac{x^6(24-32)+x^5(24-48)+x^4(-144+6-24+96)+x^3(-144-4)\ldots}{(3-x^2)^4}$$

$$+x^2(216-36+72)+x(216+12)+(54)$$

$$\frac{dy}{dx}=\frac{-8x^6-24x^5-66x^4-148x^3+252x^2+228x+54}{(3-x^2)^4}$$

e) $u(x)=x$ $u'(x)=1$ $v(x)=\sqrt{x+1}$ $v'(x)=\dfrac{1}{2\sqrt{x+1}}$

$$f'(x)=\frac{1\times\sqrt{x+1}-x\times\frac{1}{2\sqrt{x+1}}}{(\sqrt{x+1})^2}=\frac{x+1-x/2}{(x+1)\sqrt{x+1}}=\frac{x/2+1}{(x+1)\sqrt{x+1}}=\frac{x+2}{2(x+1)\sqrt{x+1}}$$

f) $y=\dfrac{x^2+2x+1}{x^2}=1+\dfrac{2}{x}+\dfrac{1}{x^2}=1+2x^{-1}+x^{-2}$

$$\frac{dy}{dx}=2\times(-1)x^{-1-1}+(-2)x^{-2-1}=-2x^{-2}-2x^{-3}$$

$$\frac{dy}{dx}=-2\left[\frac{1}{x^2}+\frac{1}{x^3}\right]=-2\left[\frac{x+1}{x^3}\right]$$

THE QUOTIENT RULE

4 $f(x)=\dfrac{\sqrt{x}}{x^2+1}$. Four steps in finding the simplest form of $f'(x)$ are given. Indicate whether each step is correct or incorrect.

A $f'(x)=\dfrac{(x^2+1)\times\frac{1}{2\sqrt{x}}-\sqrt{x}\times 2x}{(x^2+1)^2}$

B $f'(x)=\dfrac{\frac{x^2+1}{2\sqrt{x}}-2x\sqrt{x}}{(x^2+1)^2}$

C $f'(x)=\dfrac{x^2+1-4x^2}{2\sqrt{x}(x^2+1)^2}$

D $f'(x)=\dfrac{5x^2+1}{2\sqrt{x}(x^2+1)^2}$

$u(x)=\sqrt{x}$ $\quad u'(x)=1/2\sqrt{x}$

$v(x)=x^2+1$ $\quad v'(x)=2x$

$f'(x)=\dfrac{\frac{x^2+1}{2\sqrt{x}}-2x\sqrt{x}}{(x^2+1)^2}$ so $\boxed{A}$ TRUE $\boxed{B}$ TRUE too.

$f'(x)=\dfrac{x^2+1-2x\sqrt{x}\times 2\sqrt{x}}{2\sqrt{x}(x^2+1)^2}=\dfrac{x^2+1-4x^2}{2\sqrt{x}(x^2+1)^2}$ so $\boxed{C}$ TRUE

$f'(x)=\dfrac{-3x^2+1}{2\sqrt{x}(x^2+1)^2}$ $\boxed{D}$ is incorrect

5 Find the derivative of each function.

(a) $y=(x^2-4)(x+2)$ **(b)** $f(x)=4x^{\frac{5}{2}}-2x^{\frac{3}{2}}+6x^{\frac{1}{2}}$ **(c)** $y=\sqrt{5x-1}$

a) $\dfrac{dy}{dx}=2x(x+2)+1\times(x^2-4)=2x^2+4x+x^2-4=3x^2+4x-4$

Product rule

b) $f'(x)=4\times\frac{5}{2}x^{3/2}-2\times\frac{3}{2}x^{1/2}+6\times\frac{1}{2}x^{-1/2}=10x^{3/2}-3\sqrt{x}+\dfrac{3}{\sqrt{x}}$

c) $f'(x)=\dfrac{5}{2\sqrt{5x-1}}$

5 Find the derivative of each function.

(k) $y = (x^2+1)\sqrt{x}$ **(l)** $g(x) = \dfrac{\sqrt{x+1}}{\sqrt{x^2+1}}$

k) $y = x^2\sqrt{x} + \sqrt{x} = x^{5/2} + x^{1/2}$

$$\frac{dy}{dx} = \frac{5}{2}x^{\frac{5}{2}-1} + \frac{1}{2}x^{\frac{1}{2}-1} = \frac{5}{2}x^{3/2} + \frac{1}{2}x^{-1/2} = \frac{5x\sqrt{x}}{2} + \frac{1}{2\sqrt{x}}$$

l) $u(x) = \sqrt{x+1}$ $u'(x) = \dfrac{1}{2\sqrt{x+1}}$ (Chain rule)

$v(x) = \sqrt{x^2+1}$ $v'(x) = \dfrac{1}{2\sqrt{x^2+1}} \times 2x = \dfrac{x}{\sqrt{x^2+1}}$ (Chain rule)

$$g'(x) = \frac{\dfrac{\sqrt{x^2+1}}{2\sqrt{x+1}} - \dfrac{x\sqrt{x+1}}{\sqrt{x^2+1}}}{x^2+1}$$

$$g'(x) = \frac{x^2+1-2x(x+1)}{(x^2+1)2\sqrt{x+1}\sqrt{x^2+1}} = \frac{-x^2-2x+1}{2\sqrt{x+1}\,(x^2+1)^{3/2}}$$

6 Show that the gradient of the tangent to the curve $y = \dfrac{x}{x^2+1}$ is zero twice, at $x = -1$ and $x = 1$.

$$\frac{dy}{dx} = \frac{1(x^2+1) - 2x \times x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

so for $\dfrac{dy}{dx} = 0$, we must have $1 - x^2 = 0$, i.e. $x = \pm 1$