INFINITE GEOMETRIC SERIES

1 Evaluate the following: **(a)** $8 - 4 + 2 - \ldots$ **(b)** $4 + 3 + 2\frac{1}{4} + \ldots$

a) $\frac{-4}{8} = -\frac{1}{2}$ whereas $\frac{2}{-4} = -\frac{1}{2}$ so $r = -\frac{1}{2}$ (geometric series).

$S_n = \frac{8(1-(-1/2)^n)}{1-(-1/2)} = \frac{8}{3/2}\left[1-\left(-\frac{1}{2}\right)^n\right]$ and $\lim_{n\to+\infty} S_n = \frac{16}{3} = 5\frac{1}{3}$

b) $\frac{3}{4} = \frac{3}{4}$ whereas $\frac{2\,1/4}{3} = \frac{9/4}{3} = \frac{9}{12} = \frac{3}{4}$ so it's a geometric series of common ratio $3/4$, which is less than 1 (so the series converges)

$S_n = \frac{4\left[1-(3/4)^n\right]}{1-3/4} = 16\left[1-(3/4)^n\right]$ and $\lim_{n\to+\infty} S_n = 16$

1 Evaluate the following: **(c)** $25 - 10 + 4 + \ldots$ **(d)** $(\sqrt{3}+1)+1+\frac{(\sqrt{3}-1)}{2}+\ldots$

c) $\frac{-10}{25} = -\frac{2}{5}$ whereas $\frac{4}{-10} = -\frac{2}{5}$ so it's a geometric series of common ratio $(-2/5)$, which is less than 1, ∴ the series converges.

$S_n = \frac{25\left[1-(-2/5)^n\right]}{1-(-2/5)} = \frac{25}{7/5}\left[1-(-2/5)^n\right] = \frac{125}{7}\left[1-(-2/5)^n\right]$

$\therefore \lim_{n\to+\infty} S_n = \frac{125}{7} = 17\frac{6}{7}$

d) $\frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{2}$ whereas $\frac{(\sqrt{3}-1)/2}{1} = \frac{\sqrt{3}-1}{2}$ so it's a geometric series of common ratio $\left(\frac{\sqrt{3}-1}{2}\right)$, which is less than 1, ∴ the series converges

$S_n = (\sqrt{3}+1)\left[\frac{1-\left(\frac{\sqrt{3}-1}{2}\right)^n}{1-\left(\frac{\sqrt{3}-1}{2}\right)}\right] = \frac{(\sqrt{3}+1)2}{3-\sqrt{3}}\left[1-\left(\frac{\sqrt{3}-1}{2}\right)^n\right]$

So $\lim_{n\to+\infty} S_n = \frac{2(\sqrt{3}+1)(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})} = \frac{2\left[6+4\sqrt{3}\right]}{6} = \frac{6+4\sqrt{3}}{3}$

INFINITE GEOMETRIC SERIES

2 If $1 + 2x + 4x^2 + \ldots = \frac{3}{4}$, find the value of x.

$\frac{2x}{1} = 2x$ whereas $\frac{4x^2}{2x} = 2x$ so it's a geometric series of $r = 2x$, which converges, $\therefore$ we must have $2x < 1$

$$S_n = \frac{1\left[1-(2x)^n\right]}{1-2x} = \frac{1-(2x)^n}{1-2x} \quad \therefore \lim_{n\to+\infty} S_n = \frac{1}{1-2x}$$

We know that $\lim_{n\to+\infty} S_n = \frac{3}{4}$ $\quad \therefore \frac{1}{1-2x} = \frac{3}{4}$

$$\therefore 1-2x = \frac{4}{3} \iff 2x = 1-\frac{4}{3} = -\frac{1}{3}$$

$$\therefore x = -\frac{1}{6}$$

3 Find the first three terms of a geometric series given that the sum of the first four terms is $21\frac{2}{3}$ and the sum to infinity is 27.

$$T_1 + T_2 + T_3 + T_4 = 21\frac{2}{3} = S_4 = \frac{T_1\left[1-r^4\right]}{1-r} \quad \text{Eq } 1$$

Further $S_\infty = \frac{T_1}{1-r} = 27$ $\quad$ Eq 2

$\therefore$, substituting $\frac{T_1}{1-r}$ by 27 in Eq 1, we obtain

$$21\frac{2}{3} = (1-r^4)27 \iff 1-r^4 = \frac{65}{3\times 27} = \frac{65}{81}$$

$$\therefore r^4 = 1-\frac{65}{81} = \frac{16}{81} = \left(\frac{2}{3}\right)^4 \quad \therefore r = \pm\frac{2}{3}$$

* if $r = 2/3$ then $T_1 = 27\left(1-\frac{2}{3}\right) = 9$ then $T_2 = \frac{9\times 2}{3} = 6$ $\quad T_3 = \frac{6\times 2}{3} = 4$

* if $r = -2/3$ then $T_1 = 27\left(1+\frac{2}{3}\right) = 45$ then $T_2 = 45\times\left(\frac{-2}{3}\right) = -30$

and $T_3 = (-30)\times\left(\frac{-2}{3}\right) = 20$

INFINITE GEOMETRIC SERIES

5 The sum of the first four terms of a geometric series is 30 and the sum of the infinite series is 32. Find the first three terms.

$T_1 + T_2 + T_3 + T_4 = 30$ $\qquad S_\infty = 32 = \frac{T_1}{1-r}$

$S_4 = \frac{T_1(1-r^4)}{1-r} = 32(1-r^4) = 30$

so $1 - r^4 = \frac{30}{32} = \frac{15}{16}$ $\therefore r^4 = 1 - \frac{15}{16} = \frac{1}{16}$ $\therefore r = \pm\frac{1}{2}$

* if $r = \frac{1}{2}$ then $T_1 = 32\left(1-\frac{1}{2}\right) = 16$ $\quad T_2 = 8$ $\quad T_3 = 4$

* if $r = -\frac{1}{2}$ then $T_1 = 32\left(1-\left(\frac{-1}{2}\right)\right) = \frac{32\times 3}{2} = 48$

then $T_2 = -24$ and then $T_3 = 12$

7 Find the sum of the series $1 + \frac{1}{a+1} + \frac{1}{(a+1)^2} + \ldots$ For what values of a does this infinite series have a sum?

$S_n = \frac{1\left[1-\left(\frac{1}{a+1}\right)^n\right]}{1-\left(\frac{1}{a+1}\right)} = \frac{\left[1-\left(\frac{1}{a+1}\right)^n\right]}{\frac{a}{a+1}} = \frac{a+1}{a}\left[1-\left(\frac{1}{a+1}\right)^n\right]$

This sum converges only if $\left|\frac{1}{a+1}\right| < 1$, i.e. if $|a+1| > 1$

so $a > 0$ or $a < -2$

$S_\infty = \lim_{n\to+\infty} S_n = \frac{a+1}{a}$ (only if $a < -2$ or $a > 0$)

9 Find the fractional equivalent of: **(a)** $2.3\dot{8}$ **(b)** $4.\dot{6}\dot{2}$ **(c)** $0.417\,17\ldots$

a) $2.3\dot{8} = 2.38888\ldots$

$10 \times 2.3\dot{8} = 23.8888\ldots$

$100 \times 2.3\dot{8} = 238.8888$

so $(100-10) \times 2.3\dot{8} = 238 - 23 = 215$

$\therefore\ 2.3\dot{8} = \frac{215}{90} = \frac{43}{18}$

b) $4.\dot{6}\dot{2} = 4.62626262\ldots$

$100 \times 4.\dot{6}\dot{2} = 462.626262\ldots$

so $(100-1) \times 4.\dot{6}\dot{2} = 462 - 4 = 458$

$\therefore\ 4.\dot{6}\dot{2} = \frac{458}{99}$

c) $0.417\,17\,17\ldots = 0.4\dot{1}\dot{7}$

$10 \times 0.4\dot{1}\dot{7} = 4.\dot{1}\dot{7}$

$1000 \times 0.4\dot{1}\dot{7} = 417.\dot{1}\dot{7}$

$\therefore\ (1000-10) \times 0.4\dot{1}\dot{7} = 417 - 4 = 413$

$\therefore\ 0.4\dot{1}\dot{7} = \frac{413}{990}$

11 Show that 1.2888... is a rational number by expressing it in the form $\frac{m}{n}$, where m and n are integers with no common factor.

$1.2\dot{8} = 1.2888\ldots$

$10 \times 1.2\dot{8} = 12.\dot{8}$

$100 \times 1.2\dot{8} = 128.\dot{8}$

$\therefore\ (100-10) \times 1.2\dot{8} = 128 - 12 = 116$

$\therefore\ 1.2\dot{8} = \frac{116}{90} = \frac{58}{45}$

12 Evaluate $\dfrac{1+2+3+\ldots+10}{1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{512}}$. $= \dfrac{N}{D}$

$N = \dfrac{10(10+1)}{2} = 55$ using $1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$

Now for D: this is a geometric series of 1st Term 1

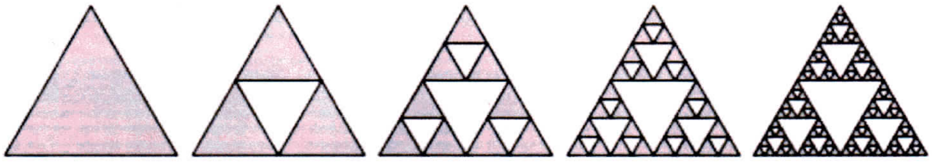
and common ratio $\left(\frac{1}{2}\right)$

$\therefore \; S_{10} = \dfrac{1\left[1-\left(\frac{1}{2}\right)^{10}\right]}{1-\frac{1}{2}} = \dfrac{1-\frac{1}{1024}}{\frac{1}{2}} = 2\left[\dfrac{1023}{1024}\right] = \dfrac{1023}{512}$

$\therefore \; \dfrac{N}{D} = \dfrac{55}{\frac{1023}{512}} = \dfrac{55 \times 512}{1023} = 27\,\dfrac{49}{93}$

INFINITE GEOMETRIC SERIES

14 The Sierpinski triangle can be constructed by starting with an equilateral triangle, dividing it into four congruent equilateral triangles and then removing the central one. The process is then repeated in each of the remaining smaller triangles as shown in the diagram.



Let the initial triangle have an area of 1 square unit.

(a) How much of the original triangle remains shaded in the third diagram?
(b) How much of the original triangle remains shaded in the tenth diagram?
(c) Discuss what happens to the remaining area as the number of triangles increases.

a) let A_n be the area of the nth triangle.

$A_1 = 1$ $\quad A_2 = \frac{3}{4} \times 1 = \frac{3}{4}$ $\quad A_3 = \frac{3}{4} \times A_2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16} \simeq 0.56$

b) $A_n = A_1 \left(\frac{3}{4}\right)^{n-1}$ $\quad \therefore \quad A_{10} = \left(\frac{3}{4}\right)^9 = \frac{19683}{262144} \simeq 0.075$

c) $\lim\limits_{n \to +\infty} A_n = \lim\limits_{n \to +\infty} \left(\frac{3}{4}\right)^{n-1} = 0$

as $\left|\frac{3}{4}\right| < 1$ therefore the remaining area theorically converges towards 0, but never reaches it.