MATHEMATICAL INDUCTION, HARDER QUESTIONS

In this section, you will construct harder mathematical induction proofs in a variety of situations. You will also study a slightly different form of induction in which you need to assume $S(1), S(2), \dots, S(k)$ are true in order to prove that $S(k+1)$ is true.

Induction question for $n > 1$

Example 17

Use mathematical induction to prove that $n^3 - n$ is a multiple of 6 for $n \geq 2$.

Solution

Step 1 $n = 2$: Exp $= 2^3 - 2 = 8 - 2 = 6$, which is a multiple of 6.

Hence the result is true for $n = 2$.

Step 2 Assume the result is true for $n = k$, i.e. assume that $k^3 - k = 6M$, where M is a positive integer.

Prove the result is true for $n = k + 1$, i.e. prove that $(k+1)^3 - (k+1)$ is a multiple of 6.

$$\begin{aligned}\text{Exp} &= (k+1)^3 - (k+1) \\ &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= 6M + 3k(k+1)\end{aligned}$$

Now since k is a positive integer then $k(k+1)$ is even so that $k(k+1) = 2N$, where N is an integer.

$$\begin{aligned}\text{Hence Exp} &= 6M + 3 \times 2N \\ &= 6(M + N) \text{ which is a multiple of 6.}\end{aligned}$$

Step 3 The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 2$, hence it is true for $n = 2 + 1$ and by the principle of mathematical induction it is true for all $n \geq 2$.

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Sigma notation

Series questions to be proved by induction are often written using sigma notation, Σ, to save space.

For example: $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n+1) = \sum_{r=1}^{n} r(r+1)$

Hence: $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + k \times (k+1) = \sum_{r=1}^{k} r(r+1)$

And: $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + k \times (k+1) + (k+1)(k+2) = \sum_{r=1}^{k+1} r(r+1)$

Example 18

Prove that $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$.

Solution

Step 1 $n = 1$: LHS $= \sum_{r=1}^{1} r(r+1) = 1 \times 2 = 2$;

RHS $= \frac{1 \times 2 \times 3}{3} = 2 =$ LHS

Result is true when $n = 1$.

Step 2 Assume the result is true for $n = k$, i.e. assume that $\sum_{r=1}^{k} r(r+1) = \frac{k(k+1)(k+2)}{3}$.

Prove the result is true for $n = k + 1$, i.e. prove that $\sum_{r=1}^{k+1} r(r+1) = \frac{(k+1)(k+2)(k+3)}{3}$.

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k+1} r(r+1) \\
&= \sum_{r=1}^{k} r(r+1) + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\
&= (k+1)(k+2)\left(\frac{k}{3} + 1\right) \\
&= (k+1)(k+2)\left(\frac{k+3}{3}\right) \\
&= \frac{(k+1)(k+2)(k+3)}{3} \\
&= \text{RHS}
\end{aligned}$$

Step 3 The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

Example 19

Prove by induction that $(2^2-1)(3^2-1)\ldots(n^2-1)=\frac{(n!)^2(n+1)}{2n}$ for all integers $n\geq 2$.

Solution

Note that this involves a product of terms rather than a sum of terms.

Let $S(n)$ be the statement that $(2^2-1)(3^2-1)\ldots(n^2-1)=\frac{(n!)^2(n+1)}{2n}$ for integers $n\geq 2$.

Step 1 Prove that $S(2)$ is true.

$$\text{LHS}=(2^2-1)=3 \qquad \text{RHS}=\frac{(2!)^2(2+1)}{2\times 2}=3$$

$$\text{LHS}=\text{RHS} \qquad \therefore S(2) \text{ is true}$$

Step 2 Assume $S(k)$ is true for a positive integer $k\geq 2$.

i.e. assume that $\boxed{(2^2-1)(3^2-1)\ldots(k^2-1)}=\frac{(k!)^2(k+1)}{2k}$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that

$$(2^2-1)(3^2-1)\ldots([k+1]^2-1)=\frac{((k+1)!)^2((k+1)+1)}{2(k+1)}$$

$$=\frac{((k+1)!)^2(k+2)}{2(k+1)}$$

$$\text{LHS}=\boxed{(2^2-1)(3^2-1)\ldots(k^2-1)}([k+1]^2-1)$$

using [a]:

$$=\frac{(k!)^2(k+1)}{2k}\times(k^2+2k)$$

$$=\frac{(k!)^2(k+1)}{2k}\times k(k+2)$$

$$=\frac{(k!)^2(k+1)(k+2)}{2}$$

$$=\frac{(k!)^2(k+1)(k+2)}{2}\times\frac{(k+1)}{(k+1)}$$

$$=\frac{((k+1)!)^2(k+2)}{2(k+1)}=\text{RHS}$$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(2)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n\geq 2$.

If the question in Example 19 had been to prove by induction that $(2^2-1)(3^2-1)\ldots(n^2-1)=\frac{(n!)^2(n+1)}{2n}$ for all integers $n\geq 1$, then the proof would not have worked.

Step 1: $n=1$, LHS $=(1^2-1)=0$

RHS $=\frac{(1!)^2\times 2}{2}=1$, which is not equal to the LHS.

Since the result cannot be proved for $n=1$ then the process of proof by mathematical induction will not work.