TANGENTS AND NORMALS TO A CURVE

1 Find the equations of the tangent and normal to the curve $y = x^2$ at $(2, 4)$.

$f(x) = x^2$ so $f'(x) = 2x$

So at $x = 2$ $f'(2) = 2 \times 2 = 4$ which is the gradient of the curve at $x = 2$

Using the gradient-point formula, the tangent is:

$y - 4 = 4(x - 2) \iff y = 4x - 4$

For the normal, the gradient is $-1/4$, so its equation is

$y - 4 = -\frac{1}{4}(x - 2) \iff y = -\frac{1}{4}x + \frac{9}{2}$

2 Find the equations of the tangent and normal to the curve $y = 3x - x^2$ at $(0, 0)$.

$f(x) = 3x - x^2$ so $f'(x) = 3 - 2x$

$f'(0) = 3$

So the equation of the tangent at $(0,0)$ is $y - 0 = 3(x - 0) \iff y = 3x$

The equation of the normal at $(0,0)$ is:

$y - 0 = -\frac{1}{3}(x - 0) \iff y = -\frac{1}{3}x$

5 Find the equations of the tangent and normal to the curve $y = 2x^2 - 4x + 1$ where the gradient is 4.

$f(x) = 2x^2 - 4x + 1$ so $f'(x) = 4x - 4$

$f'(x) = 4$ when $4x - 4 = 4 \iff 4x = 8$ so $x = 2$

At $x = 2$, $f(2) = 2 \times 2^2 - 4 \times 2 + 1 = 8 - 8 + 1 = 1$

so the point is $(2, 1)$ $f'(2) = 4$

Tangent: $y - 1 = 4(x - 2) \iff y = 4x - 7$

Normal $y - 1 = -\frac{1}{4}(x - 2) \iff y = -\frac{1}{4}x + \frac{3}{2}$

6 Find the equations of the tangent and normal to the curve $y = \frac{1}{x}$ at the point where $x = -2$. Indicate whether each statement below is a correct or incorrect step in answering this question.

(a) $\frac{dy}{dx} = \frac{1}{x^2}$

(b) At $\left(-2, -\frac{1}{2}\right)$, $\frac{dy}{dx} = -\frac{1}{4}$

(c) Equation of tangent is $x + 4y + 4 = 0$

(d) Equation of normal is $8x - 2y + 15 = 0$

At $x = -2$ $\quad f(-2) = \frac{1}{-2} = -\frac{1}{2}$ $\quad$ so Point $\left(-2, -\frac{1}{2}\right)$

$f(x) = \frac{1}{x} = x^{-1}$ $\quad$ so $f'(x) = -1 \times x^{-1-1} = -\frac{1}{x^2}$

$f'(2) = -\frac{1}{2^2} = -\frac{1}{4}$

So the equation of the tangent at $\left(-2, -\frac{1}{2}\right)$ is:

$y - \left(-\frac{1}{2}\right) = -\frac{1}{4}\left(x - (-2)\right) \iff y = -\frac{1}{4}x - \frac{1}{2} - \frac{1}{2}$

so $y = -\frac{1}{4}x - 1$ $\quad$ or $\quad 4y + x + 4 = 0$

And for the equation of the normal at that point:

$y - \left(-\frac{1}{2}\right) = 4\left(x - (-2)\right) \iff y = 4x + 8 - \frac{1}{2}$

So $y = 4x + \frac{15}{2} \iff 2y - 8x - 15 = 0$

$\iff 8x - 2y + 15 = 0$

So CORRECT = b) c) d)

INCORRECT = a)

8 Find the equations of the tangent and normal to the curve $y = 3x^3 - 7x^2 + 2x$ at the point where $x = 2$.

At $x = 2$, $f(2) = 3 \times 2^3 - 7 \times 2^2 + 2 \times 2$

$f(2) = 3 \times 8 - 7 \times 4 + 4 = 0$

so point $(2,0)$.

$f'(x) = 9x^2 - 14x + 2$

So $f'(2) = 9 \times 2^2 - 14 \times 2 + 2 = 9 \times 4 - 28 + 2 = 10$

So the equation of the tangent is, using the gradient-point formula,

$y - 0 = 10(x-2) \Leftrightarrow y = 10x - 20$

And the equation of the normal at that point is:

$y - 0 = -\frac{1}{10}(x-2)$

$\Leftrightarrow y = -\frac{1}{10}x + \frac{1}{5}$

$\Leftrightarrow x + 10y - 2 = 0$

9 The straight line $y = x + 2$ cuts the parabola $y = \frac{x^2}{2} - 2$ at two points P and Q. Find the coordinates of P and Q. Also find the equations of the tangents to the parabola at P and Q and the coordinates of the point of intersection of these tangents.

$x + 2 = \frac{x^2}{2} - 2 \iff \frac{x^2}{2} - x - 4 = 0 \iff x^2 - 2x - 8 = 0$

$\Delta = 4 - 4 \times 8 = 36$ so two roots

$x_1 = \frac{2+6}{2} = 4$ and $x_2 = \frac{2-6}{2} = \frac{-4}{2} = -2$

$\therefore$ The line $y = x + 2$ cuts the parabola $y = \frac{x^2}{2} - 2$ at

$x = 4$ and at $x = -2$

$f(4) = 4 + 2 = 6$ so point $P(4, 6)$

$f(-2) = -2 + 2 = 0$ so point $Q(-2, 0)$

$f'(x) = 2\frac{x}{2} = x$ so $f'(-2) = -2$ and $f'(4) = 4$

The equations of the tangents at P and Q are respectively:

$y - 6 = 4(x - 4)$ and $y - 0 = -2(x - (-2))$

$\iff y = 4x - 10$ $\iff y = -2x - 4$

These two lines intersect when $4x - 10 = -2x - 4$

i.e. when $6x = -4 + 10 = 6$

so at $x = 1$; when $x = 1$, $4 \times 1 - 10 = -6$

So at the point $(1, -6)$

11 Find the equations of the tangent and normal to the parabola $y = 2x^2 - 4x + 1$ at the point of zero gradient.

$f(x) = 2x^2 - 4x + 1$

$f'(x) = 4x - 4$ so $f'(x) = 0$ when $4x - 4 = 0$, i.e at $x = 1$

when $x = 1$ $f(1) = 2 \times 1^2 - 4 \times 1 + 1 = 2 - 4 + 1 = -1$

So the gradient is zero at $(1, -1)$

The equation of the tangent at that point is:

$y - (-1) = 0(x - 1) \iff y = -1$

The equation of the normal at that point is

$x = 1$

13 The line $y = x + 4$ cuts the parabola $y = x^2 - 2x$ at two points A and B. Find the size of the angles that the tangents to the curve at A and B make with the x-axis.

First we look for the coordinates of those points

The line and curve intersect when $x + 4 = x^2 - 2x$

$\Leftrightarrow x^2 - 3x - 4 = 0 \quad \Leftrightarrow (x+1)(x-4) = 0$

So at $x = -1$ and at $x = 4$

$f'(x) = 2x - 2$

$f'(-1) = 2 \times (-1) - 2 = -2 - 2 = -4$

So $\tan \theta_1 = -4 \quad \Rightarrow \theta_1 \simeq 104° 2'$

$f'(4) = 2 \times 4 - 2 = 8 - 2 = 6$

So $\tan \theta_2 = 6 \quad \Rightarrow \theta_2 = 80° 32'$

18 The line $y = x - 2$ cuts the curve $y = x^3(x - 2)$ at two points A and B. Calculate the angles that the tangents to the curve at A and B make with the x-axis and hence find the angle between the tangents.

First we look for the points which belong to both the line and the curve $y = x^3(x-2)$. ; This happens when:

$x - 2 = x^3(x-2) \iff x^3(x-2) - (x-2) = 0$

$\iff (x-2)(x^3-1) = 0$, i.e when $x = 2$, $x = 1$

if $f(x) = x^3(x-2) = x^4 - 2x^3$

then $f'(x) = 4x^3 - 6x^2$

At $x = 1$, $f'(1) = 4 \times 1^3 - 6 \times 1^2 = 4 - 6 = -2$

So $\tan\theta_1 = -2 \implies \theta_1 \simeq 116°34'$

At $x = 2$ $f'(2) = 4 \times 2^3 - 6 \times 2^2 = 4 \times 8 - 6 \times 4 = 8$

So $\tan\theta_2 = 8 \implies \theta_2 \simeq 82°52'$

So the tangents at A and B make angles of $116°34'$ and $82°52'$

The angle between the tangents is therefore:

$116°34' - 82°52' = 33°42'$

20 Find the coordinates of the points on the curve $y = x^2(2x-3)$ at which the tangent is parallel to:
(a) the line $y = 12x - 1$ **(b)** the x-axis.

a) The gradient of this line is 12

if $f(x) = x^2(2x-3) = 2x^3 - 3x^2$

then $f'(x) = 6x^2 - 6x = 6x[x-1]$

So for the gradient to be equal to 12, we must have

$6x(x-1) = 12 \iff x\cdot(x-1) = 2 \iff x^2 - x - 2 = 0$

$\Delta = 1 - 4\times(-2) = 9 = 3^2$ so two points

$x_1 = \frac{1-3}{2} = \frac{-2}{2} = -1$ and $x_2 = \frac{1+3}{2} = \frac{4}{2} = 2$

When $x_1 = -1$, then $f(-1) = (-1)^2[2\times(-1)-3] = -2-3 = -5$

So this point is $(-1, -5)$

When $x_2 = 2$ then $f(2) = 2^2(2\times 2 - 3) = 4(4-3) = 4$

so point is $(2, 4)$

b) Tangent is parallel to x-axis when gradient is 0.

For $f'(x)$ to be zero, we must have: $6x(x-1) = 0$

so $x = 0$ or $x = 1$

At $x = 0$, $f(0) = 0^2\times(2\times 0 - 3) = 0$ so point $(0,0)$

At $x = 1$, $f(1) = 1^2\times(2\times 1 - 3) = -1$ so point $(1, -1)$