SOLVING TRIGONOMETRIC EQUATIONS

Give answers correct to 3 decimal places where necessary.

1 Solve each equation for $0° \le x \le 180°$.

(a) $3 + 2\cos x = 5\cos x$ (b) $\sin x = 3\cos x$ (c) $6\sin 2x = 3\cos 30°$

a) $\Leftrightarrow 3\cos x = 3 \Leftrightarrow \cos x = 1$ so $x = 0°$

b) $\sin x = 3\cos x$

$\Leftrightarrow \tan x = 3$ $x = 71°.565$

$\Leftrightarrow x = 71°34'$

c) $6\sin 2x = 3\cos 30°$

$\Leftrightarrow 6\sin 2x = 3 \times \frac{\sqrt{3}}{2}$

$\Leftrightarrow \sin 2x = \frac{\sqrt{3}}{4}$

so $2x = 25.6589°$ i.e. $x = 12.829°$

or $2x = 180 - 25.6589 = 154.341$

so $x = 77.171°$

two solutions $x = 12.829°$ and $x = 77.171°$

SOLVING TRIGONOMETRIC EQUATIONS

Give answers correct to 3 decimal places where necessary.

1 Solve each equation for $0° \le x \le 180°$.

(d) $4 - 3\tan x = \tan x$ **(e)** $3\sin x = \cos x$ **(f)** $\sin 2x = \sin 30°$

d) $4\tan x = 4 \iff \tan x = 1$

$x = 45°$ no other solutions are possible within the interval $[0, 180°]$

e) $3\sin x = \cos x \iff \tan x = \frac{1}{3}$

$x = 18.435°$

f) $\sin 2x = \frac{1}{2}$

so either $2x = 30° \iff x = 15°$

<u>OR</u> $2x = 180 - 30 = 150°$

$\Rightarrow x = 75°$

$\therefore$ two solutions $x = 15°$ and $x = 75°$

SOLVING TRIGONOMETRIC EQUATIONS

3 Solve each equation for $0° \leq x \leq 360°$.

(a) $\operatorname{cosec}^2 x = 2$ **(b)** $\sin^2 x = 1$ **(c)** $\tan^2 x = 3$

a) $\operatorname{cosec}^2 x = 2 \iff \dfrac{1}{\sin^2 x} = 2 \iff \sin^2 x = \dfrac{1}{2}$

$\iff \sin x = \pm \dfrac{1}{\sqrt{2}} = \pm \dfrac{\sqrt{2}}{2}$

So $x = 45°$ $x = 135°$ $x = 225°$ $x = 315°$

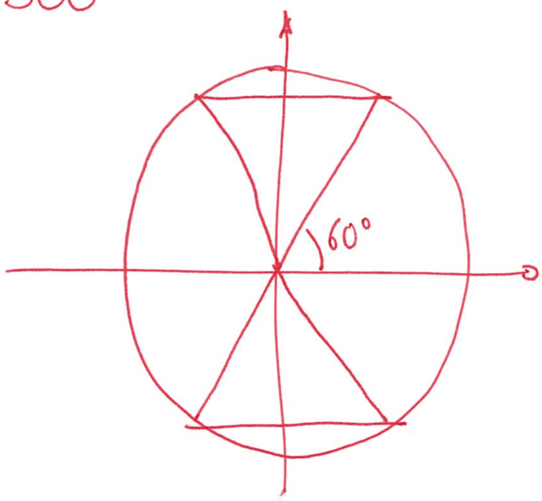
b) $\sin^2 x = 1 \iff \sin x = \pm 1$

so $x = 90°$ or $x = 270°$

c) $\tan^2 x = 3 \iff \tan x = \pm\sqrt{3} = \pm \dfrac{\sqrt{3}/2}{1/2}$

so $x = 60°$ or $x = 120°$

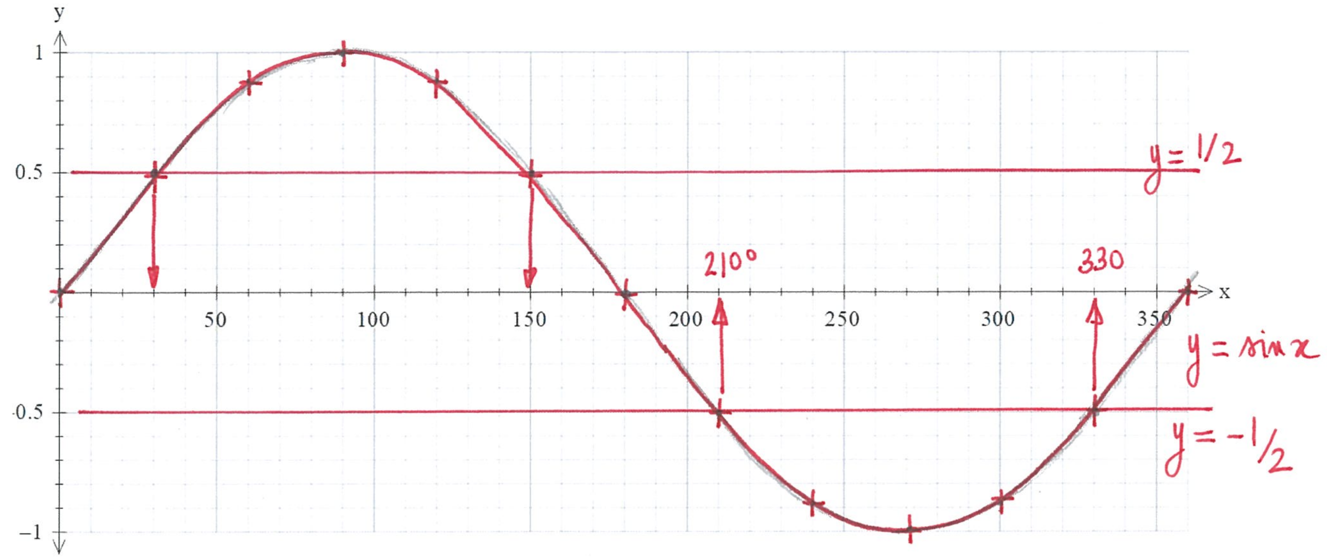
or $x = 240°$ or $x = 300°$



SOLVING TRIGONOMETRIC EQUATIONS

4 **(a)** On the same diagram draw $y = \sin x$ and $y = \frac{1}{2}$ for $0° \leq x \leq 360°$. Use your diagram to solve the equation $\sin x = \frac{1}{2}$ for $0° \leq x \leq 360°$.

(b) What line would you need to draw to solve the equation $\sin x = -\frac{1}{2}$? What are the solutions to this equation for $0° \leq x \leq 360°$?

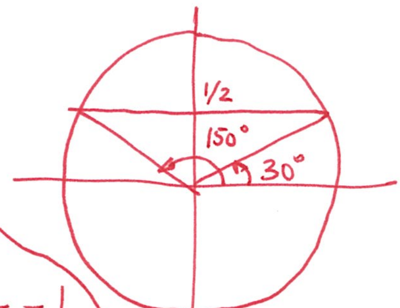


a) The line intersects the sinewave at 2 points

$x = 30°$ and $x = 150°$

At both those points $\sin x = \frac{1}{2}$

which corresponds to this situation on the unit circle



b) We need to draw the line

$y = -\frac{1}{2}$, and find the

intersections of $y = \sin x$ with the line $y = -\frac{1}{2}$

The solutions to $\sin x = -\frac{1}{2}$ are $x = 210°$ and $x = 330°$

SOLVING TRIGONOMETRIC EQUATIONS

5 Solve, for $0 \le \theta \le 2\pi$:

(a) $\sin\theta = \frac{-1}{\sqrt{2}}$ **(b)** $\sec\theta = \frac{2}{\sqrt{3}}$ **(c)** $\cot\theta = 1$

(d) $\sin^2\theta - 2\cos\theta + \cos^2\theta = 0$ **(e)** $\sin^2\theta + \cos\theta - 1 = 0$ **(f)** $\sec^2\theta - 2\tan\theta = 0$

a) $\sin\theta = -\frac{\sqrt{2}}{2}$ $\quad \theta = \frac{5\pi}{4}$ and $\theta = \frac{7\pi}{4}$

b) $\sec\theta = \frac{2}{\sqrt{3}} \iff \frac{1}{\cos\theta} = \frac{2}{\sqrt{3}} \iff \cos\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6}$ or $\theta = \frac{11\pi}{6}$

c) $\cot\theta = 1 \iff \cos\theta = \sin\theta$

$\theta = \pi/4$ or $\theta = 5\pi/4$

d) $\sin^2\theta - 2\cos\theta + \cos^2\theta = 0 \iff 1 - 2\cos\theta = 0$

$\iff \cos\theta = 1/2$

$\theta = \pi/3$ or $\theta = 5\pi/3$

e) $\sin^2\theta + \cos\theta - 1 = 0 \iff -\cos^2\theta + \cos\theta = 0$

$\iff \cos\theta\,[\cos\theta - 1] = 0$

so either $\cos\theta = 0$, i.e. $\theta = \pi/2$ or $\theta = 3\pi/2$

or $\cos\theta = 1$, i.e. $\theta = 0$ or $\theta = 2\pi$

f) $\sec^2\theta - 2\tan\theta = 0 \iff \frac{1}{\cos^2\theta} = \frac{2\sin\theta}{\cos\theta}$

$\iff \frac{1}{\cos\theta}\left[\frac{1}{\cos\theta} - 2\sin\theta\right] = 0 \iff \frac{1}{\cos\theta} = 2\sin\theta$

$\iff \sin 2\theta = 1$ as $2\sin\theta\cos\theta = \sin 2\theta$

So $2\theta = \frac{\pi}{2}$ i.e. $\theta = \frac{\pi}{4}$ OR $2\theta = \frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$

$\iff \theta = \frac{5\pi}{4}$

SOLVING TRIGONOMETRIC EQUATIONS

14 Solve for $0 < x < 2\pi$: **(a)** $5\cos^2 x + 8\sin x - 8 = 0$ **(b)** $6\tan x = 5\cot x$

a) $5\cos^2 x + 8\sin x - 8 = 0$

$\Leftrightarrow 5(1-\sin^2 x) + 8\sin x - 8 = 0$

$\Leftrightarrow -5\sin^2 x + 8\sin x - 8 + 5 = 0$

$\Leftrightarrow -5\sin^2 x + 8\sin x - 3 = 0$

We do a change of variable $X = \sin x$

$\Leftrightarrow -5X^2 + 8X - 3 = 0$ (which is a quadratic equation)

$\Delta = 8^2 - 4\times(-3)\times(-5) = 4 = 2^2$ so two solutions

$X_1 = \dfrac{-8-2}{2\times(-5)} = 1$ or $X_2 = \dfrac{-8+2}{2\times(-5)} = \dfrac{6}{10} = \dfrac{3}{5}$

For $X_1 = 1 = \sin x$ $x = \pi/2$

For $X_2 = \dfrac{3}{5} = \sin x$ $x = 0.644$ rad or $x = \pi - 0.644 = 2.498$

b) $6\tan x = 5\cot x \Leftrightarrow 6\dfrac{\sin x}{\cos x} = 5\dfrac{\cos x}{\sin x}$

$\Leftrightarrow 6\sin^2 x = 5\cos^2 x = 5(1-\sin^2 x)$

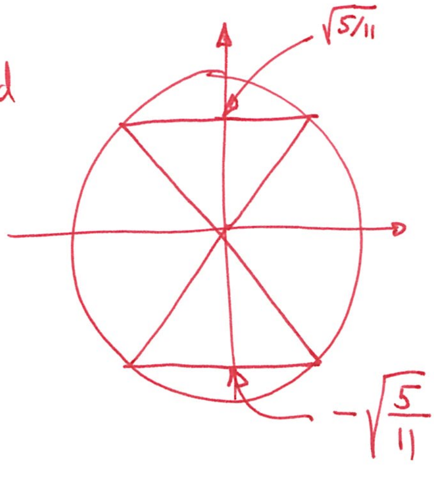
$\Leftrightarrow 11\sin^2 x = 5 \Leftrightarrow \sin^2 x = \dfrac{5}{11} \Leftrightarrow \sin x = \pm\sqrt{\dfrac{5}{11}}$

For $\sin x = \sqrt{\dfrac{5}{11}}$ gives $x = 0.740$ rad

and $x = \pi - 0.740 = 2.402$ rad

And $\sin x = -\sqrt{\dfrac{5}{11}}$ gives $x = 5.543$ rad

and $x = 3.882$ rad



15 Simplify:

(a) $1 + \tan^2(\frac{\pi}{2} - \theta)$ **(b)** $1 - \cos^2(\pi + \theta)$ **(c)** $\sin\theta\cos(\frac{\pi}{2} - \theta) + \cos\theta\sin(\frac{\pi}{2} - \theta)$

(d) $\cos^2\frac{\pi}{6} - 1$ **(e)** $1 - \sin\theta\cos(\frac{\pi}{2} - \theta)$

a) $1 + \tan^2\left(\frac{\pi}{2} - \theta\right) = 1 + \left[\frac{\sin(\pi/2 - \theta)}{\cos(\pi/2 - \theta)}\right]^2 = 1 + \left[\frac{\cos\theta}{\sin\theta}\right]^2 = 1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}$

$= \operatorname{cosec}^2\theta$

b) $1 - \cos^2(\pi + \theta) = 1 - [\cos(\pi + \theta)]^2 = 1 - [-\cos\theta]^2 = 1 - \cos^2\theta = \sin^2\theta$

c) $\sin\theta\cos\left(\frac{\pi}{2} - \theta\right) + \cos\theta\sin\left(\frac{\pi}{2} - \theta\right) = \sin\theta\sin\theta + \cos\theta\cos\theta$

$= \sin^2\theta + \cos^2\theta = 1$

d) $\cos^2\frac{\pi}{6} - 1 = \left[\frac{\sqrt{3}}{2}\right]^2 - 1 = \frac{3}{4} - 1 = \frac{3}{4} - \frac{4}{4} = -\frac{1}{4}$

e) $1 - \sin\theta\cos\left(\frac{\pi}{2} - \theta\right) = 1 - \sin\theta\sin\theta$

$= 1 - \sin^2\theta$

$= \cos^2\theta$

SOLVING TRIGONOMETRIC EQUATIONS

18 Solve for $0 \le \theta \le 2\pi$:

(a) $3\tan^3\theta - 3\tan^2\theta - \tan\theta + 1 = 0$ **(b)** $\cos^3\theta - 2\cos^2\theta + \cos\theta = 0$

a) Grouping by pairs: $\iff 3\tan^2\theta\,[\tan\theta - 1] - [\tan\theta - 1] = 0$

$\iff [\tan\theta - 1][3\tan^2\theta - 1] = 0$

So either $\tan\theta - 1 = 0$ i.e. $\theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$

OR $3\tan^2\theta - 1 = 0$ i.e $\tan^2\theta = \frac{1}{3}$, i.e. $\tan\theta = \pm\frac{1}{\sqrt{3}} = \pm\frac{1/2}{\sqrt{3}/2}$

$\theta = \frac{\pi}{6}$ $\theta = \frac{5\pi}{6}$ $\theta = \frac{7\pi}{6}$ $\theta = \frac{11\pi}{6}$

There are 6 solutions: $\frac{\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{11\pi}{6}$

b) $\cos^3\theta - 2\cos^2\theta + \cos\theta = 0 \iff \cos\theta\,[\cos^2\theta - 2\cos\theta + 1] = 0$

$\iff \cos\theta\,[\cos\theta - 1]^2 = 0$

So either $\cos\theta = 0$, i.e. $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

OR $\cos\theta = 1$ ie. $\theta = 0$ or $\theta = 2\pi$

There are 4 solutions: $0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$