APPLICATIONS INVOLVING INTEGRALS

1 A cube of ice has an edge length of 10 cm. It melts so that its volume decreases at a constant rate and the block remains a cube. If the edge length measures 5 cm after 70 minutes, find:

(a) the rate at which the volume decreases (b) the volume at any time t.

a) $\frac{dV}{dt} = C$ $V = \int C\,dt$, so $V = Ct + K$. But $V = a^3$

so $a^3 = Ct + K$ At $t=0$ $a = 10$ so $K = 10^3 = 1,000$

At $t = 70$ $a = 5$ so $125 = C \times 70 + 1000$ $C = -12.5$

So $\frac{dV}{dt} = -12.5$

b) $V = -12.5t + 1,000$

V has to be positive so $-12.5t + 1000 > 0$ $t < 80$

So $0 < t < 80$.

2 A machine manufactures items at a variable rate given by $\frac{dQ}{dt} = 2t + 1$, $t \geq 0$, where Q is the number of items manufactured in a time t minutes.

(a) At what rate is the machine working: (i) initially (ii) after 10 minutes?

(b) What is the total number of items manufactured in the first 10 minutes?

a) i) at $t = 0$ $\frac{dQ}{dt} = 1$ ii) at $t = 10$ $\frac{dQ}{dt} = 2 \times 10 + 1 = 21$

b) $Q_{10} = \int_0^{10} \frac{dQ}{dt}\,dt = \int_0^{10} (2t+1)\,dt$

$Q_{10} = \left[t^2 + t\right]_0^{10} = 100 + 10 - (0) = 110$ items.

3 The sluice gates of a dam are operated by an automatic program that controls the flow of water out of the dam. The program is set so that t hours after 7 am the flow of water will be given by

$\frac{dV}{dt} = 500 - 15t^2 + t^3$ megalitres (ML) per hour.

(a) If no water flows from the dam before 7 am, calculate:

(i) the flow of the water at 9 am

(ii) the total volume of water released between 7 am and 9 am

(b) **(i)** Sketch $\frac{dV}{dt} = 500 - 15t^2 + t^3$ for $0 \le t \le 10$.

(ii) When does the flow of water stop?

(iii) If the sluice gates close at the moment when $\frac{dV}{dt} = 0$, how much water has been released altogether?

a) i) $\frac{dV}{dt} = 500 - 15 \times 2^2 + 2^3 = 448$ ML/hr

ii) $V_{7\text{-}9} = \int_0^2 500 - 15t^2 + t^3 \, dt = \left[500t - \frac{15t^3}{3} + \frac{t^4}{4}\right]_0^2$

$= 500 \times 2 - 5 \times 2^3 + \frac{2^4}{4} = 964$ ML

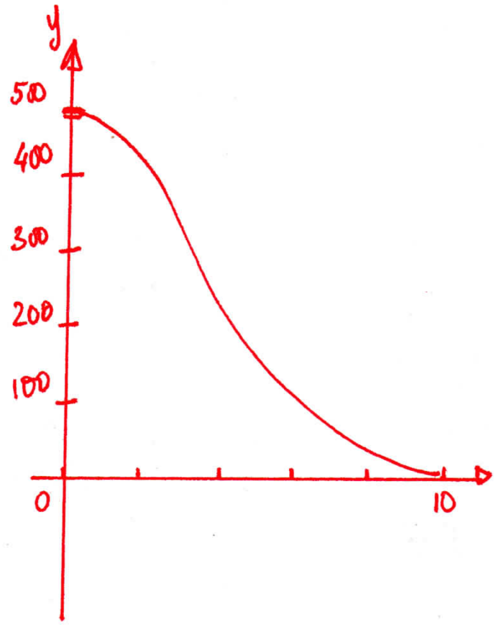
b) For $t = 10$ $\quad \frac{dV}{dt} = 500 - 15 \times 100 + 1000 = 0.$

$\frac{d^2V}{dt^2} = 0$ for $\quad -30t + 3t^2 = 0$

$\Rightarrow 3t^2 = 30 \quad t = \sqrt{10}$

At that point

$500 - 15 \times 10 + 10\sqrt{10} = 382$



ii) $7 + 10 = 17$ so 5pm

iii) $\int_0^{10} 500 - 15t^2 + t^3 \, dt = \left[500t - 5t^3 + \frac{t^4}{4}\right]_0^{10}$

$= 500 \times 10 - 5 \times 10^3 + \frac{10^4}{4}$

$= 5000 - 5000 + 2{,}500 = 2{,}500$ ML

5 A body starts from O and moves in a straight line. At any time t its velocity is given by $\dot{x} = 6t - 4$. Indicate whether each statement below is correct or incorrect.

(a) $x = 3t^2 - 4t + C$ YES with C=0 **(b)** $x = 3t^2 - 4t$ YES **(c)** $\ddot{x} = 3t^2 - 4t$ NO $\boxed{\text{(d)}}$ $\ddot{x} = 6$ correct

$\dot{x} = 6t - 4$ so $\ddot{x} = 6$

and $x(t) = \dfrac{6t^2}{2} - 4t + C = 3t^2 - 4t + C$

at $t = 0$ $x = 0$ so $C = 0$ $x = 3t^2 - 4t$

6 A body starts from O and moves in a straight line. At any time t, its velocity is $t^2 - 4t^3$. Find, in terms of t:

(a) the displacement x **(b)** the acceleration.

$\dot{x} = t^2 - 4t^3$

$x = \dfrac{t^3}{3} - 4\dfrac{t^4}{4} + C = \dfrac{t^3}{3} - t^4 + C$

$\ddot{x} = 2t - 12t^2$

At $t = 0$, $x = 0$ so $C = 0$.

$x = \dfrac{t^3}{3} - t^4$ and $\ddot{x} = 2t[1 - 6t]$

7 The velocity v m s^{-1} at time t seconds ($t \geq 0$) of a body moving in a straight line is given by $v = 6t^2 + 6t - 12$. Its initial displacement is 7 m from O. Find:

(a) the displacement and acceleration at any time t

(b) the acceleration when the velocity is zero **(c)** the initial velocity and acceleration.

a) $x = 6\dfrac{t^3}{3} + 6\dfrac{t^2}{2} - 12t + C$ $\ddot{x} = 12t + 6$

At $t = 0$ $x = 7$ so $7 = 2 \times 0^3 + 3 \times 0^2 - 12 \times 0 + C$

$C = 7$

$x = 2t^3 + 3t^2 - 12t + 7$

b) When $v = 0$, $6t^2 + 6t - 12 = 0$ or $t^2 + t - 2 = 0$

$t = 1$

At $t = 1$ $\ddot{x} = 12 \times 1 + 6 = 18$ m s^{-2}

At $t = 0$ $V = -12$ m s^{-1} $\ddot{x} = 6$ m s^{-2}

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9 A body is projected vertically upwards with an initial velocity of 30 m s^{-1}. It rises with a deceleration of 10 m s^{-2}. Find:

(a) its velocity at any time t **(b)** its height h m above the point of projection at any time t
(c) the greatest height reached **(d)** the time taken to return to the point of projection.

a) $\ddot{x} = -10$ so $\dot{x} = -10t + C$

at $t=0$ $\dot{x} = 30$ so $C = 30$ $\dot{x} = -10t + 30$

b) $x = -10\frac{t^2}{2} + 30t + K = -5t^2 + 30t + K$

At $t=0$ $x=0$ so $K=0$, i.e. $x = -5t^2 + 30t$

c) $\dot{x} = 0$ when $-10t + 30 = 0$ i.e. $t = 3$ s.

At that time $x = -5 \times 3^2 + 30 \times 3$

$x = 45$ m

d) $x = 0$ when $-5t^2 + 30t = 0$

$\Leftrightarrow 5t(-t+6) = 0$

so either $t=0$ or $t=6$.

14 Two cars A and B travel along a straight road in the same direction. Their respective distances x km from a fixed point O at any time t hours are given by the following rules:

A: $x = 50t - 20t^2$ B: $x = 80t^2 + 20t$

(a) Calculate each car's speed at the point O.

(b) At what time are the cars travelling at the same speed?

(c) Both cars reach a point Q at the same time. Calculate the distance from O to Q.

(d) A third car, travelling at uniform speed, is 2 km ahead of A and B when they pass the point O. If this car arrives at Q at the same time as A and B, find a rule connecting x and t for it.

a) For A: $\dot{x}_A = 50 - 40t$ so at $t=0$, $\dot{x}_A = \underline{50 \text{ km hr}^{-1}}$

For B $\dot{x}_B = 160t + 20$ so at $t=0$, $\dot{x}_B = \underline{20 \text{ km hr}^{-1}}$

b) $\dot{x}_A = \dot{x}_B$ when $50 - 40t = 160t + 20$

$\Leftrightarrow 200t = 30$ $t = 3/20 = 0.15$ hour or $\underline{9 \text{ minutes}}$.

c) $x_A = x_B$ when $50t - 20t^2 = 80t^2 + 20t$

$\Leftrightarrow 5 - 2t = 8t + 2$

$\Leftrightarrow 10t = 3$ $t = 3/10 = 0.3$ hour or $\underline{18 \text{ minutes}}$

At $t = 0.3$ $x = 50 \times 0.3 - 20 \times 0.3^2 = \underline{13.2 \text{ km}}$

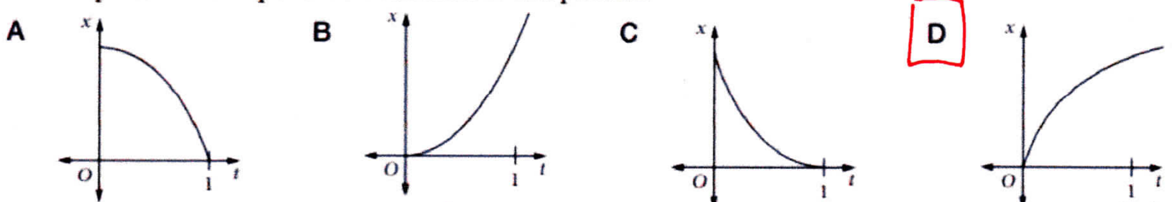
d) $\dot{x}_C = C$ So $x_C = Ct + K$

At $t=0$ $x_C = 2$ so $K = 2$ $x_C = Ct + 2$

At $t = 0.3$ $x_C = 13.2$ so $13.2 = C \times 0.3 + 2$

So $C = \dfrac{11.2}{0.3} = \dfrac{112}{3}$ $\boxed{x_C = \dfrac{112}{3}t + 2}$

16 A particle is moving so that, for $0 < t < 1$, its velocity is positive and its acceleration is negative. Which graph could represent the displacement function of this particle?



A: $v < 0$ B: $v > 0$ C: $v < 0$ D: $v > 0$

v increasing so $a > 0$

v decreasing so $a < 0$

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21 A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v.

(a) If its acceleration is $2\cos t$, and $v = 1$ and $x = 0$ when $t = 0$, find x in terms of t.

(b) If its acceleration is $-3e^{-t}$ and $v = 0$ when $t = 0$, find the time at which $v = -2$.

a) $\ddot{x} = 2\cos t$ so $\dot{x} = 2\sin t + C$

At $t=0$ $\dot{x}=1$ so $1 = 2\sin 0 + C$ $C=1$ $\dot{x} = 2\sin t + 1$

$x = -2\cos t + t + K$ At $t=0$ $x = 0 = -2 + K$ so $K = 2$

$\boxed{x = -2\cos t + t + 2}$

b) $\ddot{x} = -3e^{-t}$ $\dot{x} = 3e^{-t} + C$

At $t=0$ $v=0$ so $3e^{-0} + C = 0$ $C = -3$

$\dot{x} = 3[e^{-t} - 1]$

$\dot{x} = -2$ when $3(e^{-t} - 1) = -2$ $1 - e^{-t} = 2/3$ $e^{-t} = 1/3$

$-t = -\ln 3$ $\boxed{t = \ln 3}$

23 The acceleration of a particle moving in a straight line is given by $\frac{d^2x}{dt^2} = 12\cos 2t$. Initially $v = 0$ and $x = 6$. Find its velocity v and displacement x at any time t seconds and sketch the graph of the displacement. How many times does the particle change direction in the first 10 seconds?

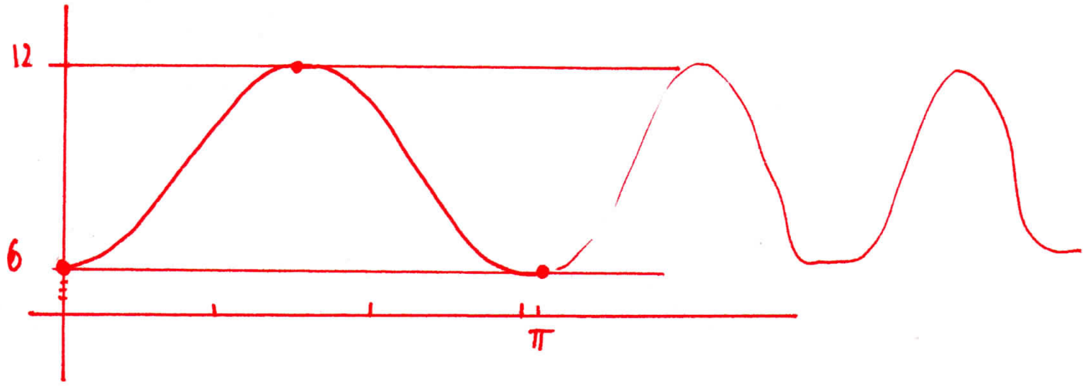
$\ddot{x} = 12\cos 2t$ so $\dot{x} = \frac{12\sin 2t}{2} + C = 6\sin 2t + C$

At $t=0$ $v=0$ so $C=0$ $\dot{x} = 6\sin 2t$

$x = -\frac{6\cos 2t}{2} + K = -3\cos 2t + K$

at $t=0$ $x=6$ so $6 = -3 + K$ $K = 9$

$x = -3\cos 2t + 9$ period π Range $[6, 12]$



$\frac{10}{\pi} = 3.2$ so 6 times