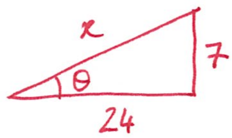
# TRIGONOMETRIC IDENTITIES AND PROOFS

**1** If $\tan\theta = \frac{7}{24}$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact value of:

**(a)** $\sin\theta$ **(b)** $\cos\theta$



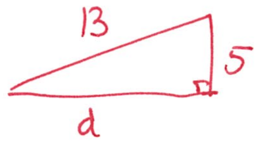

$\tan\theta = \frac{\text{opp}}{\text{adj}}$

$x^2 = 7^2 + 24^2 = 625 = 25^2$

$\pi < \theta < 3\pi/2$ so we are in the 3rd quadrant, $\therefore$ both sine and cosine are negative..

a) $\sin\theta = \frac{-7}{25}$ b) $\cos\theta = \frac{-24}{25}$

**2** If $\sin\theta = \frac{5}{13}$ and $\theta$ is acute, indicate whether each statement is correct or incorrect.




**(a)** $\cos\theta = \frac{12}{13}$ **(b)** $\sec\theta = \frac{13}{5}$ **(c)** $\tan\theta = \frac{5}{12}$ **(d)** $\cot\theta = \frac{13}{12}$

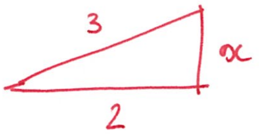
$d^2 = 13^2 - 5^2 = 144 = 12^2$ $\theta$ is acute, $\therefore$ both sine and cosine are positive

a) true b) $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{12/13} = 13/12$ so not true

c) true d) $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{12/13}{5/13} = \frac{12}{5}$ so not true.

**4** If $\cos u = \frac{2}{3}$ and $u$ is not in the first quadrant, then $\frac{\cos u - 2\cot u}{\tan u - 3\sin u}$ simplifies to:

$\boxed{\text{A}}$ $\frac{4(\sqrt{5}+6)}{15}$ **B** $\frac{5-2\sqrt{5}}{9}$ **C** $\frac{2(5-\sqrt{5})}{21}$ **D** $\frac{14}{3(4\sqrt{5}-5)}$




$3^2 = x^2 + 2^2$ so $x^2 = 9 - 4 = 5$ $x = \sqrt{5}$

$\cos u > 0$ but not in 1st quadrant, therefore it must be in the $\underline{\text{IV}}$ quadrant,

so $\sin u < 0$ and $\sin u = -\frac{\sqrt{5}}{3}$ $\tan u = \frac{-\sqrt{5}/3}{2/3} = -\frac{\sqrt{5}}{2}$

$\text{cotan}\, u = -\frac{2}{\sqrt{5}}$

$$\frac{\cos u - 2\cot u}{\tan u - 3\sin u} = \frac{\frac{2}{3} - 2\times\left(-\frac{2}{\sqrt{5}}\right)}{\left(-\frac{\sqrt{5}}{2}\right) - 3\left(-\frac{\sqrt{5}}{3}\right)} = \frac{\frac{2}{3} + \frac{4}{\sqrt{5}}}{-\frac{\sqrt{5}}{2} + \sqrt{5}} = \frac{2\sqrt{5}+12}{\frac{\sqrt{5}}{2}\times 3\sqrt{5}} = \frac{4\sqrt{5}+24}{15}$$

$$= \frac{4(\sqrt{5}+6)}{15}$$

Response $\boxed{A}$

# TRIGONOMETRIC IDENTITIES AND PROOFS

**5** Simplify:

**(a)** $\dfrac{\sin^2\theta+\cos^2\theta}{\tan^2\theta}$ **(b)** $\dfrac{\sin^2\theta}{1-\sin^2\theta}$ **(c)** $\dfrac{2\cot\alpha}{1+\cot^2\alpha}$ **(d)** $(\sec^2\theta-1)\tan(\frac{\pi}{2}-\theta)$

a) $\dfrac{\sin^2\theta+\cos^2\theta}{\tan^2\theta} = \dfrac{1}{\tan^2\theta} = \text{cotan}^2\theta$

b) $\dfrac{\sin^2\theta}{1-\sin^2\theta} = \dfrac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

c) $\dfrac{2\,\text{cotan}\,\alpha}{1+\text{cotan}^2\alpha} = \dfrac{2\,\text{cotan}\,\alpha}{\text{cosec}^2\alpha} = \dfrac{2\,\frac{\cos\alpha}{\sin\alpha}}{\left(\frac{1}{\sin\alpha}\right)^2} = \dfrac{2\cos\alpha}{\sin\alpha} \div \dfrac{1}{\sin^2\alpha}$

$= 2\,\dfrac{\cos\alpha}{\sin\alpha} \times \sin^2\alpha = 2\cos\alpha\sin\alpha = \sin 2\alpha$

d) $(\sec^2\theta-1)\tan\left(\frac{\pi}{2}-\theta\right) = \tan^2\theta \times \dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}$

(as $\sec^2\theta = 1+\tan^2\theta$)

$= \tan^2\theta \times \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta} \times \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

# TRIGONOMETRIC IDENTITIES AND PROOFS

**5** Simplify:

**(e)** $\dfrac{\sin A}{\cos A}+\dfrac{\cos A}{\sin A}$ **(f)** $\sin^3\theta+\sin\theta\cos^2\theta$ **(g)** $\operatorname{cosec}^2\theta\sin^2\theta$ **(h)** $1-\sin^2(\pi+\theta)$

e) $\dfrac{\sin A}{\cos A}+\dfrac{\cos A}{\sin A}=\dfrac{\sin^2 A+\cos^2 A}{\cos A\sin A}=\dfrac{1}{\cos A\sin A}=\dfrac{1}{\frac{\sin 2A}{2}}=\dfrac{2}{\sin 2A}$

f) $\sin^3\theta+\sin\theta\cos^2\theta=\sin\theta\left[\sin^2\theta+\cos^2\theta\right]=\sin\theta\times 1=\sin\theta$

g) $\operatorname{cosec}^2\theta\sin^2\theta=\dfrac{1}{\sin^2\theta}\times\sin^2\theta=1$

h) $1-\sin^2(\pi+\theta)=1-\left[\sin(\pi+\theta)\right]^2$

$=\cos^2(\pi+\theta)$

$=\left[\cos(\pi+\theta)\right]^2$

$=\left[-\cos\theta\right]^2$

$=\cos^2\theta$

# TRIGONOMETRIC IDENTITIES AND PROOFS

**6** Simplify:

(a) $\dfrac{x^2}{\sqrt{x^2-a^2}}$ for $x = a\sec\theta$ (b) $\sqrt{a^2-x^2}$ for $x = a\cos\theta$

a) $\dfrac{x^2}{\sqrt{x^2-a^2}} = \dfrac{a^2\sec^2\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \dfrac{a^2\sec^2\theta}{a\sqrt{\sec^2\theta - 1}} = \dfrac{a^2\sec^2\theta}{a\sqrt{\tan^2\theta}}$

$\quad = \dfrac{a^2\sec^2\theta}{a\tan\theta} = \dfrac{a\sec^2\theta}{\tan\theta}$

$\quad = a\dfrac{\frac{1}{\cos^2\theta}}{\frac{\sin\theta}{\cos\theta}} = a \times \dfrac{1}{\cos^2\theta} \div \dfrac{\sin\theta}{\cos\theta} = \dfrac{a}{\cos^2\theta} \times \dfrac{\cos\theta}{\sin\theta}$

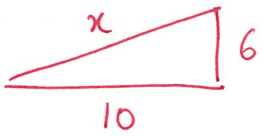
$\quad = \dfrac{a}{\sin\theta\cos\theta} = a\sec\theta\,\mathrm{cosec}\,\theta$

b) $\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\cos^2\theta} = a\sqrt{1-\cos^2\theta} = a\sqrt{\sin^2\theta}$

$\quad = a\sin\theta$

**9** Find the exact value of $\sec\theta$ if $\tan\theta = 0.6$ and $\theta$ is not in the first quadrant.

$\tan\theta > 0$ but $\theta$ is NOT in the I quadrant, so it must be in the III quadrant, i.e. both sine and cosine are negative.




$$x^2 = 6^2 + 10^2 = 136 = \left(\sqrt{136}\right)^2 = \left(\sqrt{2^3 \times 17}\right)^2 = \left(2\sqrt{34}\right)^2$$

so $$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{10}{2\sqrt{34}}} = -\frac{1}{\frac{5}{\sqrt{34}}} = -\frac{\sqrt{34}}{5}$$

**10** If $\sin\theta = x$, express $\dfrac{1-\cos^2\theta}{\sec^2\theta}$ in terms of $x$.

$$\frac{1-\cos^2\theta}{\sec^2\theta} = \frac{\sin^2\theta}{\frac{1}{\cos^2\theta}} = x^2 \div \frac{1}{\cos^2\theta} = x^2 \times \cos^2\theta$$

$$\_\_\_\_\_\_ = x^2 \times (1 - \sin^2\theta)$$

$$\_\_\_\_\_\_ = x^2 \times (1 - x^2)$$

# TRIGONOMETRIC IDENTITIES AND PROOFS

**11** If $a\sin^2\theta + b\cos^2\theta = c$, express $\sin\theta$ and $\cos\theta$ in terms of $a$, $b$ and $c$.

$\sin^2\theta + \cos^2\theta = 1$ so the equality is equivalent to $a\sin^2\theta + b(1-\sin^2\theta) = c$

$\iff \sin^2\theta\,[a-b] = c-b$ so $\sin^2\theta = \dfrac{c-b}{a-b}$

$\therefore \quad \sin\theta = \pm\sqrt{\dfrac{c-b}{a-b}}$

and $\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\dfrac{c-b}{a-b}\right) = \dfrac{a-b-(c-b)}{a-b} = \dfrac{a-c}{a-b}$

$\therefore \quad \cos\theta = \pm\sqrt{\dfrac{a-c}{a-b}}$

**13** If $\tan^2\theta + 2\sec^2\theta = 5$, find the value of $\sin^2\theta$.

$\tan^2\theta + 2\sec^2\theta = 5 \iff \dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{2}{\cos^2\theta} = 5$

$\iff \sin^2\theta + 2 = 5\cos^2\theta$

$\iff \sin^2\theta + 2 = 5(1-\sin^2\theta)$

$\iff \sin^2\theta + 2 = 5 - 5\sin^2\theta$

$\iff 6\sin^2\theta = 3$

$\iff \sin^2\theta = \dfrac{1}{2}$

# TRIGONOMETRIC IDENTITIES AND PROOFS

**14** Simplify:

**(a)** $(1+\tan^2 u)(1-\sin^2 u)$ **(b)** $\frac{1}{1-\sin V}+\frac{1}{1+\sin V}$ **(g)** $2\cos^2\frac{\pi}{6}-1$ **(h)** $1-\sin\theta\cos(\frac{\pi}{2}-\theta)$

a) $(1+\tan^2 u)(1-\sin^2 u) = \sec^2 u \times \cos^2 u = \frac{1}{\cos^2 u} \times \cos^2 u = 1$

b) $\frac{1}{1-\sin V} + \frac{1}{1+\sin V} = \frac{(1+\sin V)+(1-\sin V)}{(1-\sin V)(1+\sin V)}$

$\qquad = \frac{2}{1-\sin^2 V} = \frac{2}{\cos^2 V} = 2\sec^2 V$

g) $2\cos^2\frac{\pi}{6} - 1 = 2 \times \left(\frac{\sqrt{3}}{2}\right)^2 - 1 = 2 \times \frac{3}{4} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

h) $1 - \sin\theta \cos\left(\frac{\pi}{2} - \theta\right) = 1 - \sin\theta \times \sin\theta$

$\qquad = 1 - \sin^2\theta$

$\qquad = \cos^2\theta$

# TRIGONOMETRIC IDENTITIES AND PROOFS

Prove the following identities.

**15** $(1-\tan x)^2+(1+\tan x)^2=2\sec^2 x$

**16** $(\cot t+\operatorname{cosec} t)^2=\dfrac{1+\cos t}{1-\cos t}$

15 $(1-\tan x)^2+(1+\tan x)^2 = 1-2\tan x+\tan^2 x+1+2\tan x+\tan^2 x$

$= 2+2\tan^2 x$

$= 2(1+\tan^2 x)$

$= 2\sec^2 x$

16 $(\cot t+\operatorname{cosec} t)^2 = \left[\dfrac{\cos t}{\sin t}+\dfrac{1}{\sin t}\right]^2$

$= \left[\dfrac{\cos t+1}{\sin t}\right]^2$

$= \dfrac{[\cos t+1]^2}{\sin^2 t}$

$= \dfrac{[\cos t+1]^2}{1-\cos^2 t}$

$= \dfrac{[1+\cos t]^2}{[1-\cos t][1+\cos t]}$

$= \dfrac{1+\cos t}{1-\cos t}$

# TRIGONOMETRIC IDENTITIES AND PROOFS

Prove the following identities.

**17** $\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha - \sin^2\beta$

**18** $\sec\theta + \tan\theta = \dfrac{1+\sin\theta}{\cos\theta}$

17 $\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha\left[1-\sin^2\beta\right] - \sin^2\beta\left[1-\sin^2\alpha\right]$

$= \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta$

$= \sin^2\alpha - \sin^2\beta$

18 $\sec\theta + \tan\theta = \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}$

$= \dfrac{1+\sin\theta}{\cos\theta}$

# TRIGONOMETRIC IDENTITIES AND PROOFS

Prove the following identities.

**21** $\tan\theta(1-\cot^2\theta)+\cot\theta(1-\tan^2\theta)=0$

**22** $\dfrac{\cot\theta\cos\theta}{\cot\theta+\cos\theta}=\dfrac{\cos\theta}{1+\sin\theta}$

(21) $\tan\theta\left[1-\cot^2\theta\right]+\cot\theta\left[1-\tan^2\theta\right]=$

$$= \frac{\sin\theta}{\cos\theta}\left[1-\frac{\cos^2\theta}{\sin^2\theta}\right]+\frac{\cos\theta}{\sin\theta}\left[1-\frac{\sin^2\theta}{\cos^2\theta}\right]$$

$$= \frac{\sin\cancel{\theta}}{\cancel{\cos}\theta}-\frac{\cos\cancel{\theta}}{\cancel{\sin}\theta}+\frac{\cos\cancel{\theta}}{\cancel{\sin}\theta}-\frac{\sin\cancel{\theta}}{\cancel{\cos}\theta}=0$$

(22) $$\frac{\cot\theta\cos\theta}{\cot\theta+\cos\theta}=\frac{\frac{\cos\theta}{\sin\theta}\times\cos\theta}{\frac{\cos\theta}{\sin\theta}+\cos\theta}$$

$$= \frac{\frac{\cos^2\theta}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}+\cos\theta}$$

$$= \frac{\cos^2\theta}{\cos\theta+\sin\theta\cos\theta}$$

$$= \frac{\cos\theta}{1+\sin\theta}$$