SUM AND DIFFERENCE OF TWO ANGLES

2 Simplify:

(a) $\sin A \cos(A-B) + \cos A \sin(A-B)$
(b) $\cos(\theta+\alpha)\cos(\theta-\alpha) + \sin(\theta+\alpha)\sin(\theta-\alpha)$
(c) $\sin 2A \cos A - \cos 2A \sin A$
(d) $\cos 60° \cos 30° - \sin 60° \sin 30°$

a) We know that $\sin\alpha \cos\beta + \cos\alpha \sin\beta = \sin(\alpha+\beta)$

$\therefore \sin A \cos(A-B) + \cos A \sin(A-B) = \sin(A+A-B) = \sin(2A-B)$

b) We know that $\cos\alpha \cos\beta + \sin\alpha \sin\beta = \cos(\alpha-\beta)$

$\therefore \cos(\theta+\alpha)\cos(\theta-\alpha) + \sin(\theta+\alpha)\sin(\theta-\alpha) = \cos[(\theta+\alpha)-(\theta-\alpha)]$
$= \cos 2\alpha$

c) We know that $\sin\alpha \cos\beta - \cos\alpha \sin\beta = \sin(\alpha-\beta)$

$\therefore \sin(2A)\cos A - \cos(2A)\sin A = \sin(2A-A) = \sin A$

d) We know that $\cos\alpha \cos\beta - \sin\alpha \sin\beta = \cos(\alpha+\beta)$

$\therefore \cos 60 \cos 30 - \sin 60 \sin 30 = \cos(60+30) = \cos 90 = 0$

SUM AND DIFFERENCE OF TWO ANGLES

2 Simplify:

(e) $\frac{\tan\theta - \tan 20°}{1+\tan 20° \tan\theta}$ (f) $\frac{\tan 2\alpha + \tan\alpha}{1-\tan 2\alpha \tan\alpha}$ (g) $\sin(2A+B)\cos(A+B) - \cos(2A+B)\sin(A+B)$

(h) $\cos(3\theta+\alpha)\cos(2\theta+\alpha) + \sin(3\theta+\alpha)\sin(\theta+\alpha)$ (i) $\frac{\tan 3x - \tan x}{1+\tan 3x \tan x}$

e) We know that: $\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \tan(\alpha-\beta)$

$\therefore \frac{\tan\theta - \tan 20}{1+\tan 20 \tan\theta} = \tan(\theta - 20)$

f) We know that $\frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \tan(\alpha+\beta)$

$\therefore \frac{\tan 2\alpha + \tan\alpha}{1-\tan 2\alpha \tan\alpha} = \tan(2\alpha+\alpha) = \tan 3\alpha$

g) $\sin(2A+B)\cos(A+B) - \cos(2A+B)\sin(A+B) = \sin[(2A+B)-(A+B)]$

$= \sin[A]$

h) $\cos(3\theta+\alpha)\cos(2\theta+\alpha) + \sin(3\theta+\alpha)\sin(\theta+\alpha) = \cos[(3\theta+\alpha)-(2\theta+\alpha)]$

$= \cos\theta$

i) $\frac{\tan 3x - \tan x}{1+\tan 3x\tan x} = \tan(3x-x) = \tan 2x$

SUM AND DIFFERENCE OF TWO ANGLES

4 (a) Find the exact value of $\sin 38^\circ \cos 22^\circ + \cos 38^\circ \sin 22^\circ$.

(b) Find the exact value of $\dfrac{\tan 119^\circ + \tan 16^\circ}{1 - \tan 119^\circ \tan 16^\circ}$.

(c) Find the exact value of $\cos 165^\circ$.

(d) Expand and simplify $\sin(x + 40^\circ) + \sin(x - 40^\circ)$.

a) $\sin 38 \cos 22 + \cos 38 \sin 22 = \sin(38 + 22)$

$= \sin 60 = \sqrt{3}/2$

b) $\dfrac{\tan 119 + \tan 16}{1 - \tan 119 \tan 16} = \tan(119 + 16) = \tan(135)$

$= \dfrac{\sin(135)}{\cos(135)} = \dfrac{\sqrt{2}/2}{\left(-\sqrt{2}/2\right)} = -1$

c) $\cos 165 = \cos(120 + 45)$

$= \cos 120 \cos 45 - \sin 120 \sin 45$

$= \left(-\dfrac{1}{2}\right) \times \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{2}}{2}$

$= -\dfrac{\sqrt{2}}{4} - \dfrac{\sqrt{6}}{4} = -\left(\dfrac{\sqrt{2} + \sqrt{6}}{4}\right)$

d) $\sin(x + 40) + \sin(x - 40) = [\sin x \cos 40 + \cos x \sin 40]$

$+ [\sin x \cos 40 - \cos x \sin 40]$

$= 2 \sin x \cos 40$

6 Write the expansion of $\cos(\theta-\phi)$. Write $(90°-\theta)$ in place of θ to deduce the expansion of $\sin(\theta+\phi)$.

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$\sin(\theta+\phi) = \cos[90-(\theta+\phi)] = \cos[(90-\theta)-\phi]$$

$$= \cos(90-\theta)\cos\phi + \sin(90-\theta)\sin\phi$$

$$= \sin\theta\cos\phi + \cos\theta\sin\phi$$

7 If θ and ϕ are angles between 0° and 90°, $\sin\theta = \frac{3}{5}$, $\tan\phi = \frac{7}{24}$, find the following without using a calculator.

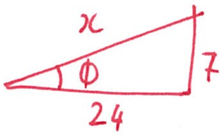
(a) $\sin(\theta-\phi)$ **(b)** $\cos(\theta+\phi)$ **(c)** $\tan(\theta-\phi)$

a) $\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

$\sin\theta = 3/5$ and $0<\theta<90$ (so $\cos\theta>0$) so $\cos^2\theta = 1-\sin^2\theta = 1-\left(\frac{3}{5}\right)^2$

$\cos\theta = \frac{4}{5}$

$\tan\phi = \frac{7}{24}$



so $x^2 = 7^2+24^2 = 625$ $x=25$

So $\sin\phi = 7/25$ and $\cos\phi = \frac{24}{25}$

$$\therefore \sin(\theta-\phi) = \frac{3}{5}\times\frac{24}{25} - \frac{4}{5}\times\frac{7}{25} = \frac{72-28}{125} = \frac{44}{125}$$

b) $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$$= \frac{4}{5}\times\frac{24}{25} - \frac{3}{5}\times\frac{7}{25} = \frac{96-21}{125} = \frac{3}{5}$$

c) $$\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi} = \frac{\frac{3}{4}-\frac{7}{24}}{1+\frac{3}{4}\times\frac{7}{24}}$$

$$\tan(\theta-\phi) = \frac{11/24}{39/32} = \frac{11}{24}\div\frac{39}{32} = \frac{11}{24}\times\frac{32}{39} = \frac{352}{936} = \frac{44}{117}$$

SUM AND DIFFERENCE OF TWO ANGLES

10 (a) Using the expansion of $\sin(A+B)$, prove that $\sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}$.

(b) Using the expansion of $\tan(A+B)$, prove that $\tan 75° = 2+\sqrt{3}$.

a) $\sin 75 = \sin(30+45) = \sin 30 \cos 45 + \cos 30 \sin 45$

$$\text{——} = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

b) $$\tan 75 = \tan(30+45) = \frac{\tan 30 + \tan 45}{1 - \tan 30 \tan 45}$$

$$\therefore \tan 75 = \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}} \times 1}$$

$$\therefore \tan 75 = \frac{1+\sqrt{3}}{\sqrt{3}-1} = \frac{(1+\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$ (we rationalise the denominator)

$$\tan 75 = \frac{1+3+2\sqrt{3}}{3-1}$$

$$\tan 75 = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$$

11 Find the value (in simplest surd form) of:

(a) $\cos 75°$ **(b)** $\tan 15°$ **(c)** $\cos 15°$

a) $\cos 75 = \cos(30+45) = \cos 30 \cos 45 - \sin 30 \sin 45$

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

b) $\tan 15 = \tan(45-30) = \frac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30}$

$\tan 15 = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$\tan 15 = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{3+1-2\sqrt{3}}{3-1} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$

c) $\cos 15 = \cos(45-30) = \cos 45 \cos 30 + \sin 45 \sin 30$

$\cos 15 = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$

$\cos 15 = \frac{\sqrt{6}+\sqrt{2}}{4}$