DERIVATIVE OF COMPOSITION OF TWO FUNCTIONS ("chain rule")

If $h(x) = f\big(g(x)\big)$ then $h'(x) = f'\big(g(x)\big) \times g'(x)$

or noted in a simplified way: $[f(g)]' = f'(g) \times g'$

Note: $f\big(g(x)\big)$ is often noted $f \circ g(x)$

Proof:

Let $h(x) = f\big(g(x)\big)$ By definition: $h'(x) = \lim_{h\to 0} \frac{h(x+h)-h(x)}{h}$

$$h'(x) = \lim_{h\to 0}\left[\frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h}\right]$$

$$h'(x) = \lim_{h\to 0}\left[\frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{g(x+h)-g(x)} \times \frac{g(x+h)-g(x)}{h}\right]$$

$$h'(x) = \lim_{h\to 0}\left[\frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{g(x+h)-g(x)}\right] \times \lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

Now we do a change of variable, as follows: $g(x+h) - g(x) = k$

and therefore: $g(x+h) = g(x) + k$

We also note that if $h \to 0$, then $g(x+h) \to g(x)$

$$r'(x) = \lim_{k\to 0}\left[\frac{f\big(g(x)+k\big) - f\big(g(x)\big)}{k}\right] \times \lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

Therefore: $\left[f\big(g(x)\big)\right]' = f'\big(g(x)\big) \times g'(x)$

or noted in a simplified way: $[f(g)]' = f'(g) \times g'$ referred to as the "*chain rule*"

Example: if $h(x) = \sqrt{x^2+3x-9}$ let: $f(u) = \sqrt{u}$ and $g(x) = x^2 + 3x - 9$

then : $f'(u) = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$ and $g'(x) = 2x + 3$

and therefore: $h'(x) = \frac{1}{2\sqrt{x^2+3x-9}} \times (2x+3)$ which simplifies as: $h'(x) = \frac{2x+3}{2\sqrt{x^2+3x-9}}$